Measure of Similarity between Interval-Valued Fuzzy Numbers Based on Standard Deviation Operator

Shi-Jay Chen, Hsiao-Wei Kao

***Abstract*— In decision-making processes, evaluations of parameters or variables often involve real-world problems represented by interval-valued fuzzy numbers. This study proposes a novel similarity measurement method based on the standard deviation operator to address limitations in existing approaches for measuring similarity between interval-valued fuzzy numbers. The theoretical properties of the proposed similarity measure are systematically demonstrated. To validate its effectiveness, the proposed method is compared with existing similarity measures using 17 sets of interval-valued fuzzy numbers. The comparison results show that the proposed method outperforms existing methods in terms of accuracy and applicability.**

***Index Terms*— Interval-Valued Fuzzy Numbers Similarity Measure, Standard Deviation.**

I. Introduction

Interval-valued fuzzy numbers (IVFNs) play a critical role in various domains due to their capacity to represent uncertainty. Guijun et al. [16] comprehensively described IVFNs and extended their operations, while Wang and Li [19] introduced the correlation coefficient for IVFNs and discussed its properties. Lin [18] utilized IVFNs to model vague processing times in job-shop scheduling problems, and Yao and Lin [18] applied IVFNs to construct a fuzzy flow-shop sequencing model with unknown job processing times. Hong and Lee [17] developed a distance measure for IVFNs, and Wei and Chen [21] proposed a similarity measure for IVFNs to address fuzzy risk analysis problems. Collectively, these studies highlight the versatility of IVFNs in solving real-world problems.

Numerous methods have been proposed to measure the degree of similarity between IVFNs [6], [8], [10], [12], [21]. However, existing similarity measures often exhibit limitations, such as their inability to accurately assess the similarity between certain IVFNs under specific conditions. To address these issues, this study introduces a novel similarity measure for IVFNs based on the standard deviation operator. The proposed measure overcomes key limitations in existing methods and demonstrates improved accuracy and reliability.

The properties of the proposed similarity measure are systematically analyzed, and its performance is compared against five established methods [6], [8], [10], [12], [21] using 17 sets of IVFNs. Comparative results reveal that the proposed measure effectively addresses the shortcomings of existing approaches, further underscoring its potential for broader applications.

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II. Preliminaries

We briefly reviews the definitions of the generalized trapezoidal fuzzy number [2], [3], the interval-valued fuzzy set [14], the interval-valued fuzzy number [19] and some existing similarity measures of interval-valued fuzzy numbers [6], [8], [10], [12], [21].

Chen [2], [3] definitions a generalized trapezoidal fuzzy number by $\tilde{A}=(a_1,a_2,a_3,a_4; w_{\tilde{A}})$, where $0 < w_{\tilde{A}} \le 1$, $0 \le a_1 \le a_2 \le a_3 \le a_4 \le 1$, and a_1, a_2, a_3, and a_4 denote real numbers. Chen and Chen [4] presented the Simple Center of Gravity Method (SCGM) to calculate the COG points of generalized trapezoidal fuzzy numbers. Assume that there is an generalized trapezoidal fuzzy number $\tilde{A} = (a_1,a_2,a_3,a_4; w_{\tilde{A}})$, the two values $y^*_{\tilde{A}}$ and $x^*_{\tilde{A}}$ of the COG points of generalized fuzzy number $\tilde{A}$ calculated as follows:

$$y_{\tilde{A}}^* = \begin{cases} \dfrac{w_{\tilde{A}} \times \left(\dfrac{a_3-a_2}{a_4-a_1}+2\right)}{6}, & \text{if } a_1 \ne a_4 \text{ and } 0 < w_{\tilde{A}} \le 1, \\ \dfrac{w_{\tilde{A}}}{2}, & \text{if } a_1 = a_4 \text{ and } 0 < w_{\tilde{A}} \le 1, \end{cases} \tag{1}$$

$$x^*_{\tilde{A}} = \frac{y^*_{\tilde{A}}(a_3+a_2)+(a_4+a_1)(w_{\tilde{A}}-y^*_{\tilde{A}})}{2w_{\tilde{A}}}. \tag{2}$$

Hong and Lee [17] indicate that an interval-valued fuzzy set C defined in the universe of discourse X by

$$C=\{(x,[\mu_C^L(x), \mu_C^U(x)]) \mid x \in X\},$$

where $0 \le \mu_C^L(x) \le \mu_C^U(x) \le 1$ and the membership grade $\mu_C(x)$ of an element x belongs to the interval-valued fuzzy set C is represented by an interval $\mu_C(x) = [\mu_C^L(x), \mu_C^U(x)]$, where $0 \le \mu_C^L(x) \le \mu_C^U(x) \le 1$ and the membership grade $\mu_C(x)$ of an element x belongs to the interval-valued fuzzy set C is represented by an interval $\mu_C(x) = [\mu_C^L(x), \mu_C^U(x)]$.

Yao and Lin [22] pointed out that if $\tilde{\tilde{A}}^L = (a_1^L, a_2^L, a_3^L, a_4^L; w_{\tilde{\tilde{A}}^L})$ and $\tilde{\tilde{A}}^U = (a_1^U, a_2^U, a_3^U, a_4^U; w_{\tilde{\tilde{A}}^U})$, where $a_1^L \le a_2^L \le a_3^L \le a_4^L$, $a_1^U \le a_2^U \le a_3^U \le a_4^U$, $0 \le w_{\tilde{\tilde{A}}^L} \le 1$, $0 < w_{\tilde{\tilde{A}}^U} \le 1$, and $\tilde{\tilde{A}}^L \subset \tilde{\tilde{A}}^U$, then the interval-valued trapezoidal fuzzy number $\tilde{\tilde{A}} = [\tilde{\tilde{A}}^L, \tilde{\tilde{A}}^U] = [(a_1^L, a_2^L, a_3^L, a_4^L; w_{\tilde{\tilde{A}}^L}), (a_1^U, a_2^U, a_3^U, a_4^U; w_{\tilde{\tilde{A}}^U})]$,as shown in Figure. 1.

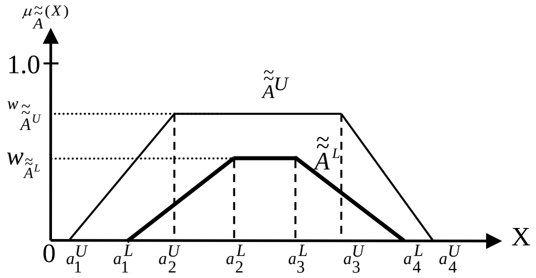

Figure.1. Interval-valued trapezoidal fuzzy number $\tilde{\tilde{A}}$.

Figure. 1 displays that $\tilde{\tilde{A}}^L$ and $\tilde{\tilde{A}}^U$ denoted as two elements of the interval-valued trapezoidal fuzzy number $\tilde{\tilde{A}}$, where $\tilde{\tilde{A}}^L$ is named "lower trapezoidal fuzzy number", and $\tilde{\tilde{A}}^U$ is named "upper trapezoidal fuzzy number". From Figure.1, the two elements $\tilde{\tilde{A}}^L$ and $\tilde{\tilde{A}}^U$ of the interval-valued trapezoidal fuzzy number $\tilde{\tilde{A}}$ can be regarded as two different generalized fuzzy numbers $\tilde{\tilde{A}}^L$ and $\tilde{\tilde{A}}^U$, where $\tilde{\tilde{A}}^L=(a_1^L, a_2^L, a_3^L, a_4^L; w_{\tilde{\tilde{A}}^L})$, $\tilde{\tilde{A}}^U=(a_1^U, a_2^U, a_3^U, a_4^U; w_{\tilde{\tilde{A}}^U})$.If $a_1^L=a_1^U$, $a_2^L=a_2^U$, $a_3^L=a_3^U$, $a_4^L=a_4^U$ and $w_{\tilde{\tilde{A}}}^L=w_{\tilde{\tilde{A}}^U}=w_{\tilde{\tilde{A}}}$, then the interval-valued trapezoidal fuzzy number $\tilde{\tilde{A}}$ can be regarded as a generalized trapezoidal fuzzy number, denoted as $\tilde{\tilde{A}}=(a_1, a_2, a_3, a_4; w_{\tilde{\tilde{A}}})$.

Chen and Chen [6] presented a similarity measure between interval-valued trapezoidal fuzzy numbers. Consider two interval-valued trapezoidal fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$. The degree of similarity $S(\tilde{\tilde{A}},\tilde{\tilde{B}})$ between the interval-valued trapezoidal fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ can be calculated as follows:

$$S(\tilde{\tilde{A}},\tilde{\tilde{B}})=\sqrt{S(\tilde{\tilde{A}}^L,\tilde{\tilde{B}}^L)\times S(\tilde{\tilde{A}}^U,\tilde{\tilde{B}}^U)}\,, \tag{3}$$

where $S(\tilde{\tilde{A}}^L,\tilde{\tilde{B}}^L)$ and $S(\tilde{\tilde{A}}^U,\tilde{\tilde{B}}^U)$ are calculated by formulae (4) and (5),

$$S(\tilde{\tilde{A}}^L,\tilde{\tilde{B}}^L)=\left[(1-\sum_{i=1}^{4}\left|a_i^L-b_i^L\right|\Big/4)\times(1-\left|x^*_{\tilde{\tilde{A}}^L}-x^*_{\tilde{\tilde{B}}^L}\right|)\right]^{1/2}\times\frac{\min(y^*_{\tilde{\tilde{A}}^L},y^*_{\tilde{\tilde{B}}^L})}{\max(y^*_{\tilde{\tilde{A}}^L},y^*_{\tilde{\tilde{B}}^L})}, \tag{4}$$

$$S(\tilde{\tilde{A}}^U,\tilde{\tilde{B}}^U)=\left[(1-\sum_{i=1}^{4}\left|a_i^U-b_i^U\right|\Big/4)\times(1-\left|x^*_{\tilde{\tilde{A}}^U}-x^*_{\tilde{\tilde{B}}^U}\right|)\right]^{1/2}\times\frac{\min(y^*_{\tilde{\tilde{A}}^U},y^*_{\tilde{\tilde{B}}^U})}{\max(y^*_{\tilde{\tilde{A}}^U},y^*_{\tilde{\tilde{B}}^U})}, \tag{5}$$

where $i=1,2,3,4$, $S(\tilde{\tilde{A}},\tilde{\tilde{B}})\in[0,1]$, $x^*_{\tilde{\tilde{A}}^L}$ and $y^*_{\tilde{\tilde{A}}^L}$ denote the COG point of the lower trapezoidal fuzzy number. According to formulas (1) and (2), $x^*_{\tilde{\tilde{A}}^L}$ and $y^*_{\tilde{\tilde{A}}^L}$ can be calculated as follows:

$$y^*_{\tilde{\tilde{A}}^L}=\begin{cases}\dfrac{w_{\tilde{\tilde{A}}^L}\times\left(\dfrac{a_3^L-a_2^L}{a_4^L-a_1^L}+2\right)}{6}, & \text{if } a_1^L\neq a_4^L \text{ and } 0<w_{\tilde{\tilde{A}}^L}\leq 1,\\[2ex] \dfrac{w_{\tilde{\tilde{A}}^L}}{2}, & \text{if } a_1^L=a_4^L \text{ and } 0<w_{\tilde{\tilde{A}}^L}\leq 1,\end{cases} \tag{6}$$

$$x^*_{\tilde{\tilde{A}}^L}=\frac{y^*_{\tilde{\tilde{A}}^L}(a_3^L+a_2^L)+(a_4^L+a_1^L)(w_{\tilde{\tilde{A}}^L}-y^*_{\tilde{\tilde{A}}^L})}{2w_{\tilde{\tilde{A}}^L}} \tag{7}$$

In the same way, $x^*_{\tilde{\tilde{A}}^U}$ and $y^*_{\tilde{\tilde{A}}^U}$ can be calculated by formulas (1) and (2). The larger the value of $S(\tilde{\tilde{A}},\tilde{\tilde{B}})$, the greater the similarity between the interval-valued fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$.

Chen [8] presented a similarity measure between interval-valued trapezoidal fuzzy numbers. The degree of similarity $S(\tilde{\tilde{A}},\tilde{\tilde{B}})$between the interval-valued trapezoidal fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ can be calculated as follows:

$$S(\tilde{\tilde{A}},\tilde{\tilde{B}})=\sqrt{S(\tilde{\tilde{A}}^L,\tilde{\tilde{B}}^L)\times S(\tilde{\tilde{A}}^U,\tilde{\tilde{B}}^U)}\,, \tag{8}$$

where $S(\tilde{\tilde{A}}^L,\tilde{\tilde{B}}^L)$ and $S(\tilde{\tilde{A}}^U,\tilde{\tilde{B}}^U)$ are calculated by formulae (9) and (10),

$$S(\tilde{\tilde{A}}^L,\tilde{\tilde{B}}^L)=\left[\sqrt[4]{\prod_{i=1}^{4}(2-\left|a_i^L-b_i^L\right|)}-1\right]\times\frac{\min(y^*_{\tilde{\tilde{A}}^L},y^*_{\tilde{\tilde{B}}^L})}{\max(y^*_{\tilde{\tilde{A}}^L},y^*_{\tilde{\tilde{B}}^L})}, \tag{9}$$

$$S(\tilde{\tilde{A}}^U,\tilde{\tilde{B}}^U)=\left[\sqrt[4]{\prod_{i=1}^{4}(2-\left|a_i^U-b_i^U\right|)}-1\right]\times\frac{\min(y^*_{\tilde{\tilde{A}}^U},y^*_{\tilde{\tilde{B}}^U})}{\max(y^*_{\tilde{\tilde{A}}^U},y^*_{\tilde{\tilde{B}}^U})}, \tag{10}$$

where $S(\tilde{\tilde{A}}^L,\tilde{\tilde{B}}^L)\in[0,1]$, $S(\tilde{\tilde{A}}^U,\tilde{\tilde{B}}^U)\in[0,1]$, and $S(\tilde{\tilde{A}},\tilde{\tilde{B}})\in[0,1]$, $i=1,2,3,4$.. The values $y^*_{\tilde{\tilde{A}}^L}$, $y^*_{\tilde{\tilde{A}}^U}$, $y^*_{\tilde{\tilde{B}}^L}$, and $y^*_{\tilde{\tilde{B}}^U}$ are calculated by formula (1). The larger the value of $S(\tilde{\tilde{A}},\tilde{\tilde{B}})$, the greater the similarity between the interval-valued fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$.

Wei and Chen [21] presented a similarity measure between interval-valued trapezoidal fuzzy numbers. Their method combines the concepts of geometric distance, the perimeter, the height and the COG points of interval-valued fuzzy numbers to calculate the degree of similarity between interval-valued trapezoidal fuzzy numbers. The degree of similarity $S(\tilde{\tilde{A}},\tilde{\tilde{B}})$between the interval-valued trapezoidal fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ can be calculated as follows:

$$S(\tilde{\tilde{A}},\tilde{\tilde{B}})=\left[\frac{S(\tilde{\tilde{A}}^L,\tilde{\tilde{B}}^L)+S(\tilde{\tilde{A}}^U,\tilde{\tilde{B}}^U)}{2}\times(1-\Delta x)\times(1-\Delta y)\right]^{\left(\frac{1}{1+2t}\right)}\times\left(1-\left|w_{\tilde{\tilde{A}}^U}-w_{\tilde{\tilde{B}}^U}-w_{\tilde{\tilde{A}}^L}+w_{\tilde{\tilde{B}}^L}\right|\right)^{\frac{u}{2}}, \tag{11}$$

where $s(\tilde{\tilde{A}}^L,\tilde{\tilde{B}}^L) \in [0, 1]$, $s(\tilde{\tilde{A}}^U,\tilde{\tilde{B}}^U) \in [0, 1]$, and $S(\tilde{\tilde{A}},\tilde{\tilde{B}}) \in [0, 1]$. The values $s(\tilde{\tilde{A}}^L,\tilde{\tilde{B}}^L)$, $s(\tilde{\tilde{A}}^U,\tilde{\tilde{B}}^U)$, Δx and Δy can be calculated as follows:

$$s(\tilde{\tilde{A}}^L,\tilde{\tilde{B}}^L)=\begin{cases}\left[1-\dfrac{\sum_{i=1}^{4}\left|a_i^L-b_i^L\right|}{4}\right] \times\dfrac{\min\left(L(\tilde{\tilde{A}}^L),L(\tilde{\tilde{B}}^L)\right)+\min(w_{\tilde{\tilde{A}}^L},w_{\tilde{\tilde{B}}^L})}{\max\left(L(\tilde{\tilde{A}}^L),L(\tilde{\tilde{B}}^L)\right)+\max(w_{\tilde{\tilde{A}}^L},w_{\tilde{\tilde{B}}^L})}, & if\ \min(w_{\tilde{\tilde{A}}^L},w_{\tilde{\tilde{B}}^L})\neq 0\\ 0, & otherwise\end{cases} \tag{12}$$

$$s(\tilde{\tilde{A}}^U,\tilde{\tilde{B}}^U)=\begin{cases}\left[1-\dfrac{\sum_{i=1}^{4}\left|a_i^U-b_i^U\right|}{4}\right] \times\dfrac{\min\left(L(\tilde{\tilde{A}}^U),L(\tilde{\tilde{B}}^U)\right)+\min(w_{\tilde{\tilde{A}}^U},w_{\tilde{\tilde{B}}^U})}{\max\left(L(\tilde{\tilde{A}}^U),L(\tilde{\tilde{B}}^U)\right)+\max(w_{\tilde{\tilde{A}}^U},w_{\tilde{\tilde{B}}^U})}, & if\ \min(w_{\tilde{\tilde{A}}^U},w_{\tilde{\tilde{B}}^U})\neq 0\\ 0, & otherwise\end{cases} \tag{13}$$

$$\Delta x=\begin{cases}\left|x^*_{\tilde{\tilde{A}}}-x^*_{\tilde{\tilde{B}}}\right|, & if\ A(\tilde{\tilde{A}}^U)-A(\tilde{\tilde{A}}^L)\neq 0\ and\ A(\tilde{\tilde{B}}^U)-A(\tilde{\tilde{B}}^L)\neq 0\\ 0, & otherwise\end{cases} \tag{14}$$

$$\Delta y=\begin{cases}\left|y^*_{\tilde{\tilde{A}}}-y^*_{\tilde{\tilde{B}}}\right|, & if\ A(\tilde{\tilde{A}}^U)-A(\tilde{\tilde{A}}^L)\neq 0\ and\ A(\tilde{\tilde{B}}^U)-A(\tilde{\tilde{B}}^L)\neq 0\\ 0, & otherwise\end{cases} \tag{15}$$

where

$$L(\tilde{\tilde{A}}^L)=\sqrt{(a_1^L-a_2^L)^2+w^2_{\tilde{\tilde{A}}^L}}+\sqrt{(a_3^L-a_4^L)^2+w^2_{\tilde{\tilde{A}}^L}}+(a_3^L-a_2^L)+(a_4^L-a_1^L). \tag{16}$$

In the same way, the values $L(\tilde{\tilde{A}}^U)$, $L(\tilde{\tilde{B}}^L)$, and $L(\tilde{\tilde{B}}^U)$ can be calculated using formula (16).The values $A(\tilde{\tilde{A}}^L)$ and $A(\tilde{\tilde{A}}^U)$ denote the areas of the lower trapezoidal fuzzy number $\tilde{\tilde{A}}^L$ and the upper trapezoidal fuzzy number $\tilde{\tilde{A}}^U$, and be calculated as follows:

$$A(\tilde{\tilde{A}}^U)=\frac{(a_3^U-a_2^U)+(a_4^U-a_1^U)}{2}\times w_{\tilde{\tilde{A}}^U}, \tag{17}$$

$$A(\tilde{\tilde{A}}^L)=\frac{(a_3^L-a_2^L)+(a_4^L-a_1^L)}{2}\times w_{\tilde{\tilde{A}}^L}. \tag{18}$$

In the same way, the values $A(\tilde{\tilde{B}}^L)$ and $A(\tilde{\tilde{B}}^U)$can be calculated using formulas (17) and (18). The values $x^*_{\tilde{\tilde{A}}}$ and $y^*_{\tilde{\tilde{A}}}$ denote the COG point of the interval-valued fuzzy number $\tilde{\tilde{A}}$, and are calculated as follows:

$$x^*_{\tilde{\tilde{A}}}=\begin{cases}\dfrac{A(\tilde{\tilde{A}}^U)x^*_{\tilde{\tilde{A}}^U}-A(\tilde{\tilde{A}}^L)x^*_{\tilde{\tilde{A}}^L}}{A(\tilde{\tilde{A}}^U)-A(\tilde{\tilde{A}}^L)}, & if\ A(\tilde{\tilde{A}}^U)-A(\tilde{\tilde{A}}^L)\neq 0\\ 0, & otherwise\end{cases} \tag{19}$$

$$y^*_{\tilde{\tilde{A}}}=\begin{cases}\dfrac{A(\tilde{\tilde{A}}^U)y^*_{\tilde{\tilde{A}}^U}-A(\tilde{\tilde{A}}^L)y^*_{\tilde{\tilde{A}}^L}}{A(\tilde{\tilde{A}}^U)-A(\tilde{\tilde{A}}^L)}, & if\ A(\tilde{\tilde{A}}^U)-A(\tilde{\tilde{A}}^L)\neq 0\\ 0, & otherwise\end{cases} \tag{20}$$

The values $x^*_{\tilde{\tilde{A}}^U}$, $x^*_{\tilde{\tilde{A}}^L}$, $y^*_{\tilde{\tilde{A}}^U}$, $y^*_{\tilde{\tilde{A}}^L}$, $A(\tilde{\tilde{A}}^U)$, and $A(\tilde{\tilde{A}}^L)$ are calculated by formulas (6), (7), (17), and (18). Similarly, the two values $x^*_{\tilde{\tilde{B}}}$ and $y^*_{\tilde{\tilde{B}}}$ denote the COG point of the interval-valued fuzzy number $\tilde{\tilde{B}}$ can be calculated as formulas (19) and (20). The larger the value of $S(\tilde{\tilde{A}},\tilde{\tilde{B}})$, the greater the similarity between the interval-valued fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$.

Chen and Chen [12] presented a similarity measure between interval-valued trapezoidal fuzzy numbers. This method considers the similarity of the gravities on the X-axis between upper fuzzy numbers, the difference of the spreads between upper fuzzy numbers, the heights of the upper fuzzy numbers, the degree of similarity on the X-axis between interval-valued fuzzy numbers, and the gravities on the Y-axis between interval-valued fuzzy numbers. The degree of similarity $S(\tilde{\tilde{A}},\tilde{\tilde{B}})$ between the interval-valued trapezoidal fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ can be calculated as follows:

$$S(\tilde{\tilde{A}},\tilde{\tilde{B}})=\frac{S_X^U(\tilde{\tilde{A}}^U,\tilde{\tilde{B}}^U)\times\left(1-\left|w_{\tilde{\tilde{A}}^U}-w_{\tilde{\tilde{B}}^U}\right|\right)}{1+STD^U(\tilde{\tilde{A}}^U,\tilde{\tilde{B}}^U)}\times S_X(\tilde{\tilde{A}},\tilde{\tilde{B}})\times S_Y(\tilde{\tilde{A}},\tilde{\tilde{B}}), \tag{21}$$

where $S(\tilde{\tilde{A}},\tilde{\tilde{B}}) \in [0, 1]$. $S_X^U(\tilde{\tilde{A}}^U,\tilde{\tilde{B}}^U)$, $S_X(\tilde{\tilde{A}},\tilde{\tilde{B}})$, $S_Y(\tilde{\tilde{A}},\tilde{\tilde{B}})$, $STD^U(\tilde{\tilde{A}}^U,\tilde{\tilde{B}}^U)$ can be calculated as follows:

$$S_X^U(\tilde{\tilde{A}}^U,\tilde{\tilde{B}}^U)=1-\frac{\sum_{i=1}^{4}\left|a_i^U-b_i^U\right|}{4}, \tag{22}$$

$$S_X(\tilde{\tilde{A}},\tilde{\tilde{B}})=1-\frac{\sum_{i=1}^{4}\left|(a_i^U-a_i^L)-(b_i^U-b_i^L)\right|}{4}, \tag{23}$$

$$S_Y(\tilde{\tilde{A}},\tilde{\tilde{B}})=1-\left|y_{\tilde{\tilde{A}}}-y_{\tilde{\tilde{B}}}\right|, \tag{24}$$

$$STD^U(\tilde{\tilde{A}}^U,\tilde{\tilde{B}}^U)=\left|STD_{\tilde{\tilde{A}}^U}-STD_{\tilde{\tilde{B}}^U}\right|, \tag{25}$$

where $S_X^U(\tilde{\tilde{A}}^U,\tilde{\tilde{B}}^U) \in [0, 1]$, $S_X(\tilde{\tilde{A}},\tilde{\tilde{B}}) \in [0,1]$, $S_Y(\tilde{\tilde{A}},\tilde{\tilde{B}}) \in [0, 1]$, i =1,2,3,4.. The value $y_{\tilde{\tilde{A}}}$ is calculated as follows:

$$y_{\tilde{\tilde{A}}}=\begin{cases}w_{\tilde{\tilde{A}}^U}, & ,if\ \tilde{\tilde{A}}^U=\tilde{\tilde{A}}^L\\ \dfrac{y_{\tilde{\tilde{A}}^U}\times A(\tilde{\tilde{A}}^U)-y_{\tilde{\tilde{A}}^L}\times A(\tilde{\tilde{A}}^L)}{A(\tilde{\tilde{A}}^U)-A(\tilde{\tilde{A}}^L)}, & ,if\ \tilde{\tilde{A}}^U\neq\tilde{\tilde{A}}^L\end{cases} \tag{26}$$

The values $y_{\tilde{\tilde{A}}^U}$, $y_{\tilde{\tilde{A}}^L}$, $A(\tilde{\tilde{A}}^U)$, and $A(\tilde{\tilde{A}}^L)$ are calculated by formulas (1), (17), and (18).In the same way, the value $y_{\tilde{\tilde{B}}}$ can be calculated using formula (26). The value $STD_{\tilde{\tilde{A}}^U}$ denotes the standard deviation of the upper trapezoidal fuzzy number $\tilde{\tilde{A}}^U$:

$$STD_{\tilde{\tilde{A}}^U} = \sqrt{\frac{\sum_{i=1}^{4}(a_i^U - \bar{x}_{\tilde{\tilde{A}}^U})^2}{4-1}}, \tag{27}$$

where

$$\bar{x}_{\tilde{\tilde{A}}^U} = \frac{\sum_{i=1}^{4} a_i^U}{4}. \tag{28}$$

where i =1, 2, 3, 4. In the same way, the value $STD_{\tilde{\tilde{B}}^U}$ can be calculated as formulas (27) and (28). The larger the value of $S(\tilde{\tilde{A}}, \tilde{\tilde{B}})$, the greater the similarity between $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$.

Chen [10] proposed a similarity measure for interval-valued trapezoidal fuzzy numbers using the geometric-mean operator. This method calculates the distance values Δa_i on the X-axis between the lower and upper bounds of trapezoidal fuzzy numbers $\tilde{\tilde{A}}^L$ and $\tilde{\tilde{A}}^U$ for each interval-valued trapezoidal fuzzy number $\tilde{\tilde{A}}$. Subsequently, it determines the degree of similarity $S(\tilde{\tilde{A}}^\Delta, \tilde{\tilde{B}}^\Delta)$ based on the calculated distance values Δa_i and Δb_i of the two interval-valued trapezoidal fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$. The overall degree of similarity $S(\tilde{\tilde{A}}, \tilde{\tilde{B}})$ between the interval-valued trapezoidal fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ .is computed as follows:

$$S(\tilde{\tilde{A}}, \tilde{\tilde{B}}) = \frac{S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U) \times \left[1 + S(\tilde{\tilde{A}}^\Delta, \tilde{\tilde{B}}^\Delta)\right]}{2} \tag{29}$$

where $S(\tilde{\tilde{A}}, \tilde{\tilde{B}}) \in [0, 1]$. $S(\tilde{\tilde{A}}^\Delta, \tilde{\tilde{B}}^\Delta)$ and $S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U)$ can be calculated as follows:

$$S(\tilde{\tilde{A}}^\Delta, \tilde{\tilde{B}}^\Delta) = \left[\sqrt[4]{\prod_{i=1}^{4}(2 - \left|\Delta a_i - \Delta b_i\right|)} - 1\right] \times \left(1 - \left|w_{\tilde{\tilde{A}}^L} - w_{\tilde{\tilde{B}}^L}\right|\right) \tag{30}$$

where i =1,2,3,4 and $S(\tilde{\tilde{A}}^\Delta, \tilde{\tilde{B}}^\Delta) \in [0, 1]$. Calculate the distance values Δa_i on the X-axis between the lower and upper trapezoidal fuzzy numbers $\tilde{\tilde{A}}^L$ and $\tilde{\tilde{A}}^U$ of the interval-valued trapezoidal fuzzy number $\tilde{\tilde{A}}$ shown as follows:

$$\Delta a_i = \left|a_i^U - a_i^L\right|, \tag{31}$$

where i =1,2,3,4. In the same way, the distance values Δb_i on the X-axis between the lower and upper trapezoidal fuzzy numbers $\tilde{\tilde{B}}^L$ and $\tilde{\tilde{B}}^U$ of the interval-valued trapezoidal fuzzy number $\tilde{\tilde{B}}$ can be calculated as formulas (31). The value $S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U)$ denotes the degree of similarity between the upper trapezoidal fuzzy numbers $\tilde{\tilde{A}}^U$ and $\tilde{\tilde{B}}^U$ as follows:

$$S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U) = \left[\sqrt[4]{\prod_{i=1}^{4}(2 - \left|a_i^U - b_i^U\right|)} - 1\right] \times \frac{\min(w_{\tilde{\tilde{A}}^U}, w_{\tilde{\tilde{B}}^U})}{\max(w_{\tilde{\tilde{A}}^U}, w_{\tilde{\tilde{B}}^U})}, \tag{32}$$

where $S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U) \in [0, 1]$, where i =1,2,3,4. The larger the value of $S(\tilde{\tilde{A}}, \tilde{\tilde{B}})$ the greater the similarity between interval-valued trapezoidal fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$.

III. New Method for Calculating the Degree of Similarity Between Interval-Valued Trapezoidal Fuzzy Numbers

In this section, we propose a new similarity measure calculate the degree of similarity between interval-valued trapezoidal fuzzy numbers, and we explicate some properties of the proposed method. Let U be the universe of discourse, U= [0, 1]. Consider two interval-valued trapezoidal fuzzy numbers $\tilde{\tilde{A}} = [\tilde{\tilde{A}}^L, \tilde{\tilde{A}}^U] = [(a_1^L, a_2^L, a_3^L, a_4^L; w_{\tilde{\tilde{A}}^L}), (a_1^U, a_2^U, a_3^U, a_4^U; w_{\tilde{\tilde{A}}^U})]$ and $\tilde{\tilde{B}} = [\tilde{\tilde{B}}^L, \tilde{\tilde{B}}^U] = [(b_1^L, b_2^L, b_3^L, b_4^L; w_{\tilde{\tilde{B}}^L}), (b_1^U, b_2^U, b_3^U, b_4^U; w_{\tilde{\tilde{B}}^U})]$, where $0 \le a_1^L \le a_2^L \le a_3^L \le a_4^L \le 1$, $0 \le a_1^U \le a_2^U \le a_3^U \le a_4^U \le 1$, $0 \le w_{\tilde{\tilde{A}}^L} \le 1$, $0 < w_{\tilde{\tilde{A}}^U} \le 1$ and $\tilde{\tilde{A}}^L \subset \tilde{\tilde{A}}^U$; $0 \le b_1^U \le b_2^U \le b_3^U \le b_4^U \le 1$, $0 \le b_1^L \le b_2^L \le b_3^L \le b_4^L \le 1$, $0 \le w_{\tilde{\tilde{B}}^L} \le w_{\tilde{\tilde{B}}^U} \le 1$ and $\tilde{\tilde{B}}^L \subset \tilde{\tilde{B}}^U$. The degree of similarity between interval-valued trapezoidal fuzzy numbers can be calculated for the proposed method presented as follows:

Step1: Calculate the distance values Δa_i on the X-axis between the lower and upper trapezoidal fuzzy numbers $\tilde{\tilde{A}}^L$ and $\tilde{\tilde{A}}^U$ of the interval-valued trapezoidal fuzzy number $\tilde{\tilde{A}}$ shown as follows:

$$\Delta a_i = \left|a_i^U - a_i^L\right|, \tag{33}$$

where i =1,2,3,4. In the same way, the distance values Δb_i on the X-axis between the lower and upper trapezoidal fuzzy numbers $\tilde{\tilde{B}}^L$ and $\tilde{\tilde{B}}^U$ of the interval-valued trapezoidal fuzzy number $\tilde{\tilde{B}}$ can be calculated as formula (33).

Step2: Calculate the degree of similarity $S(\tilde{\tilde{A}}^\Delta, \tilde{\tilde{B}}^\Delta)$ between the distance values Δa_i and Δb_i of the two interval-valued trapezoidal fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ as follows:

$$S(\tilde{\tilde{A}}^\Delta, \tilde{\tilde{B}}^\Delta) = \left[1 - \frac{\sqrt{\sum_{i=1}^{4}(\Delta a_i - \Delta b_i)^2}}{2}\right] \times \left[1 - \sqrt{\frac{\left|\Delta S_a - \Delta S_b\right|}{2}}\right] \times \left[1 - \frac{\left|w_{\tilde{\tilde{A}}^L} - w_{\tilde{\tilde{B}}^L}\right|}{\left|w_{\tilde{\tilde{A}}^U} + w_{\tilde{\tilde{B}}^U}\right|}\right], \tag{34}$$

where i =1, 2, 3, 4 and $S(\tilde{\tilde{A}}^\Delta, \tilde{\tilde{B}}^\Delta) \in [0, 1]$ and ΔS_a denotes standard deviation between the lower and the upper trapezoidal fuzzy number $\tilde{\tilde{A}}^L$ and $\tilde{\tilde{A}}^U$ of interval-valued trapezoidal fuzzy number $\tilde{\tilde{A}}$ shown as follows:

$$\Delta S_a = \left| S_{\tilde{\tilde{A}}^U} - S_{\tilde{\tilde{A}}^L} \right|, \quad (35)$$

where $S_{\tilde{\tilde{A}}^U}$ and $S_{\tilde{\tilde{A}}^L}$ can be calculated as follows:

$$S_{\tilde{\tilde{A}}^U} = \sqrt{\frac{\sum_{i=1}^{4}(a_i^U - \bar{a}^U)}{n-1}}, \quad (36)$$

$$S_{\tilde{\tilde{A}}^L} = \sqrt{\frac{\sum_{i=1}^{4}(a_i^L - \bar{a}^L)}{n-1}}, \quad (37)$$

where $\bar{a}^U$ denotes average of upper trapezoidal fuzzy number $\tilde{\tilde{A}}^U$, and $\bar{a}^L$ denotes average of lower trapezoidal fuzzy number $\tilde{\tilde{A}}^L$. In the same way, the value ΔS_b can be calculated as formulas (35), (36) and (37).

Step3: Calculate the degree of similarity $S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U)$ between the upper trapezoidal fuzzy numbers $\tilde{\tilde{A}}^U$ and $\tilde{\tilde{B}}^U$ as follows:

$$S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U) = \left[1 - \frac{\sqrt{\sum_{i=1}^{4}\left(a_i^U - b_i^U\right)^2}}{2}\right] \times \left[1 - \sqrt{\frac{\left|S_{\tilde{\tilde{A}}^U} - S_{\tilde{\tilde{B}}^U}\right|}{2}}\right] \times \frac{\min(w_{\tilde{\tilde{A}}^U}, w_{\tilde{\tilde{B}}^U})}{\max(w_{\tilde{\tilde{A}}^U}, w_{\tilde{\tilde{B}}^U})}, \quad (38)$$

where $S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U) \in [0, 1]$.

Step4: Calculate the degree of similarity $S(\tilde{\tilde{A}}, \tilde{\tilde{B}})$ between the interval-valued trapezoidal fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ as follows:

$$S(\tilde{\tilde{A}}, \tilde{\tilde{B}}) = \frac{S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U) \times (1 + S(\tilde{\tilde{A}}^\Delta, \tilde{\tilde{B}}^\Delta))}{2}, \quad (39)$$

where $S(\tilde{\tilde{A}}, \tilde{\tilde{B}}) \in [0, 1]$.The larger the value of $S(\tilde{\tilde{A}}, \tilde{\tilde{B}})$ the greater the similarity between interval-valued trapezoidal fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$.

The proposed similarity measure between interval-valued trapezoidal fuzzy numbers has the following properties:

Property 3.1: Two interval-valued trapezoidal fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ .are identical if and only if $S(\tilde{\tilde{A}}, \tilde{\tilde{B}})$ =1.

Proof:

(i)If $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ .are identical interval-valued trapezoidal fuzzy numbers, then

$a_1^L = b_1^L$, $a_2^L = b_2^L$, $a_3^L = b_3^L$, $a_4^L = b_4^L$, $a_1^U = b_1^U$, $a_2^U = b_2^U$, $a_3^U = b_3^U$, $a_4^U = b_4^U$, $w_{\tilde{\tilde{A}}^L} = w_{\tilde{\tilde{B}}^L}$ and $w_{\tilde{\tilde{A}}^U} = w_{\tilde{\tilde{B}}^U}$.

According to formula (33), the distance values Δa_i and Δb_i between the two interval-valued trapezoidal fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ are identical, i.e. $\Delta a_i = \Delta b_i$, where i =1, 2, 3, 4. According to formula (34), (35), (36) and (37), the degree of similarity $S(\tilde{\tilde{A}}^\Delta, \tilde{\tilde{B}}^\Delta)$between the distance values Δa_i and Δb_i of the two interval-valued trapezoidal fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$, and the degree of similarity $S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U)$ between the upper trapezoidal fuzzy numbers $\tilde{\tilde{A}}^U$ and $\tilde{\tilde{B}}^U$ can be calculated as follows:

$$S(\tilde{\tilde{A}}^\Delta, \tilde{\tilde{B}}^\Delta) = \left[1 - \frac{\sqrt{\sum_{i=1}^{4}(\Delta a_i - \Delta b_i)^2}}{2}\right] \times \left[1 - \sqrt{\frac{|\Delta S_a - \Delta S_b|}{2}}\right] \times \left[1 - \frac{\left|w_{\tilde{\tilde{A}}^L} - w_{\tilde{\tilde{B}}^L}\right|}{\left|w_{\tilde{\tilde{A}}^U} + w_{\tilde{\tilde{B}}^U}\right|}\right]$$

$$= \left[1 - \frac{\sqrt{0}}{2}\right] \times \left[1 - \sqrt{\frac{0}{2}}\right] \times \left[1 - \frac{|0|}{\left|w_{\tilde{\tilde{A}}^U} + w_{\tilde{\tilde{B}}^U}\right|}\right]$$

$$= 1 \times 1 \times 1 = 1.$$

$$S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U) = \left[1 - \frac{\sqrt{\sum_{i=1}^{4}\left(a_i^U - b_i^U\right)^2}}{2}\right] \times \left[1 - \sqrt{\frac{\left|S_{\tilde{\tilde{A}}^U} - S_{\tilde{\tilde{B}}^U}\right|}{2}}\right] \times \frac{\min(w_{\tilde{\tilde{A}}^U}, w_{\tilde{\tilde{B}}^U})}{\max(w_{\tilde{\tilde{A}}^U}, w_{\tilde{\tilde{B}}^U})}$$

$$= \left[1 - \frac{\sqrt{0}}{2}\right] \times \left[1 - \sqrt{\frac{0}{2}}\right] \times 1$$

$$=1.$$

Therefore,

$$S(\tilde{\tilde{A}}, \tilde{\tilde{B}}) = \frac{S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U) \times (1 + S(\tilde{\tilde{A}}^\Delta, \tilde{\tilde{B}}^\Delta))}{2}$$

$$=1.$$

(ii)If $S(\tilde{\tilde{A}}, \tilde{\tilde{B}}) = 1$, then $S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U) = 1$ and $S(\tilde{\tilde{A}}^\Delta, \tilde{\tilde{B}}^\Delta) = 1$.

According to formula (38), if $S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U) = 1$, then

$$S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U) = \left[1 - \frac{\sqrt{\sum_{i=1}^{4}\left(a_i^U - b_i^U\right)^2}}{2}\right] \times \left[1 - \sqrt{\frac{\left|S_{\tilde{\tilde{A}}^U} - S_{\tilde{\tilde{B}}^U}\right|}{2}}\right] \times \frac{\min(w_{\tilde{\tilde{A}}^U}, w_{\tilde{\tilde{B}}^U})}{\max(w_{\tilde{\tilde{A}}^U}, w_{\tilde{\tilde{B}}^U})}$$

$$=1.$$

It's express that $a_i^U = b_i^U$, $S_{\tilde{\tilde{A}}^U} = S_{\tilde{\tilde{B}}^U}$, $w_{\tilde{\tilde{A}}^U} = w_{\tilde{\tilde{B}}^U}$, where i =1, 2, 3, 4. Hence, the upper trapezoidal fuzzy numbers $\tilde{\tilde{A}}^U$ and $\tilde{\tilde{B}}^U$ are identical. According to formula (34), (35), (36) and (37), we knows that if $S(\tilde{\tilde{A}}^\Delta, \tilde{\tilde{B}}^\Delta) = 1$, then

$$S(\tilde{\tilde{A}}^\Delta, \tilde{\tilde{B}}^\Delta) = \left[1 - \frac{\sqrt{\sum_{i=1}^{4}(\Delta a_i - \Delta b_i)^2}}{2}\right] \times \left[1 - \sqrt{\frac{|\Delta S_a - \Delta S_b|}{2}}\right] \times \left[1 - \frac{\left|w_{\tilde{\tilde{A}}^L} - w_{\tilde{\tilde{B}}^L}\right|}{\left|w_{\tilde{\tilde{A}}^U} + w_{\tilde{\tilde{B}}^U}\right|}\right]$$

$$=1.$$

It's express that $\Delta a_i = \Delta b_i$, $\Delta S_a = \Delta S_b$ and $w_{\tilde{\tilde{A}}^L} = w_{\tilde{\tilde{B}}^L}$, where i =1, 2, 3, 4. According to formula (33), because Δa_i is equal to

Δb_i and a_i^U is equal to b_i^U, a_i^L must be equal to b_i^L, where i =1, 2, 3, 4.

Based on (i) and (ii), the interval-valued trapezoidal fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ are identical.

Property 3.2: $S(\tilde{\tilde{A}}, \tilde{\tilde{B}}) = S(\tilde{\tilde{B}}, \tilde{\tilde{A}})$.

proof: Based on formula (39),

$$S(\tilde{\tilde{A}}, \tilde{\tilde{B}}) = \frac{S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U)\times(1+S(\tilde{\tilde{A}}^\Delta, \tilde{\tilde{B}}^\Delta))}{2},$$

$$S(\tilde{\tilde{B}}, \tilde{\tilde{A}}) = \frac{S(\tilde{\tilde{B}}^U, \tilde{\tilde{A}}^U)\times(1+S(\tilde{\tilde{B}}^\Delta, \tilde{\tilde{A}}^\Delta))}{2},$$

where

$$S(\tilde{\tilde{A}}^\Delta, \tilde{\tilde{B}}^\Delta) = \left[1-\frac{\sqrt{\sum_{i=1}^{4}(\Delta a_i-\Delta b_i)^2}}{2}\right]\times\left[1-\sqrt{\frac{|\Delta S_a-\Delta S_b|}{2}}\right]\times\left[1-\frac{|w_{\tilde{\tilde{A}}^L}-w_{\tilde{\tilde{B}}^L}|}{|w_{\tilde{\tilde{A}}^U}+w_{\tilde{\tilde{B}}^U}|}\right],$$

$$S(\tilde{\tilde{B}}^\Delta, \tilde{\tilde{A}}^\Delta) = \left[1-\frac{\sqrt{\sum_{i=1}^{4}(\Delta b_i-\Delta a_i)^2}}{2}\right]\times\left[1-\sqrt{\frac{|\Delta S_b-\Delta S_a|}{2}}\right]\times\left[1-\frac{|w_{\tilde{\tilde{B}}^L}-w_{\tilde{\tilde{A}}^L}|}{|w_{\tilde{\tilde{B}}^U}+w_{\tilde{\tilde{A}}^U}|}\right],$$

$$S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U) = \left[1-\frac{\sqrt{\sum_{i=1}^{4}\left(a_i^U-b_i^U\right)^2}}{2}\right]\times\left[1-\sqrt{\frac{|S_{\tilde{\tilde{A}}^U}-S_{\tilde{\tilde{B}}^U}|}{2}}\right]\times\frac{\min(w_{\tilde{\tilde{A}}^U}, w_{\tilde{\tilde{B}}^U})}{\max(w_{\tilde{\tilde{A}}^U}, w_{\tilde{\tilde{B}}^U})},$$

$$S(\tilde{\tilde{B}}^U, \tilde{\tilde{A}}^U) = \left[1-\frac{\sqrt{\sum_{i=1}^{4}\left(b_i^U-a_i^U\right)^2}}{2}\right]\times\left[1-\sqrt{\frac{|S_{\tilde{\tilde{B}}^U}-S_{\tilde{\tilde{A}}^U}|}{2}}\right]\times\frac{\min(w_{\tilde{\tilde{B}}^U}, w_{\tilde{\tilde{A}}^U})}{\max(w_{\tilde{\tilde{B}}^U}, w_{\tilde{\tilde{A}}^U})},$$

where $(\Delta a_i-\Delta b_i)^2=(\Delta b_i-\Delta a_i)^2$, $\left(a_i^U-b_i^U\right)^2=\left(b_i^U-a_i^U\right)^2$, $|\Delta S_a-\Delta S_b|=|\Delta S_b-\Delta S_a|$, $|S_{\tilde{\tilde{A}}^U}-S_{\tilde{\tilde{B}}^U}|=|S_{\tilde{\tilde{B}}^U}-S_{\tilde{\tilde{A}}^U}|$, $|w_{\tilde{\tilde{A}}^L}-w_{\tilde{\tilde{B}}^L}|=|w_{\tilde{\tilde{B}}^L}-w_{\tilde{\tilde{A}}^L}|$, $|w_{\tilde{\tilde{A}}^U}+w_{\tilde{\tilde{B}}^U}|=|w_{\tilde{\tilde{B}}^U}+w_{\tilde{\tilde{A}}^U}|$, $\frac{\min(w_{\tilde{\tilde{A}}^U}, w_{\tilde{\tilde{B}}^U})}{\max(w_{\tilde{\tilde{A}}^U}, w_{\tilde{\tilde{B}}^U})}=\frac{\min(w_{\tilde{\tilde{B}}^U}, w_{\tilde{\tilde{A}}^U})}{\max(w_{\tilde{\tilde{B}}^U}, w_{\tilde{\tilde{A}}^U})}$ and i =1, 2, 3, 4.. Hence, $S(\tilde{\tilde{A}}^L, \tilde{\tilde{B}}^L) = S(\tilde{\tilde{B}}^L, \tilde{\tilde{A}}^L)$ and $S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U) = S(\tilde{\tilde{B}}^U, \tilde{\tilde{A}}^U)$.

Property 3.3: If $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ be real values between zero and one. Where $\tilde{\tilde{A}} = a$ and $\tilde{\tilde{B}} = b$, then $S(\tilde{\tilde{A}}, \tilde{\tilde{B}}) = 1 - |a - b|$.

Proof : If $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ be real values, then

$\tilde{\tilde{A}} = [(a_1^L, a_2^L, a_3^L, a_4^L; w_{\tilde{\tilde{A}}^L}), (a_1^U, a_2^U, a_3^U, a_4^U; w_{\tilde{\tilde{A}}^U})]$
$= [(a, a, a, a; 1), (a, a, a, a; 1)]$
$= (a, a, a, a; 1)$
$= a$,

$\tilde{\tilde{B}} = [(b_1^L, b_2^L, b_3^L, b_4^L; w_{\tilde{\tilde{B}}^L}), (b_1^U, b_2^U, b_3^U, b_4^U; w_{\tilde{\tilde{B}}^U})]$
$= [(b, b, b, b; 1), (b, b, b, b; 1)]$
$= (b, b, b, b; 1)$
$= b$.

When $w_{\tilde{\tilde{A}}^L} = w_{\tilde{\tilde{B}}^L} = w_{\tilde{\tilde{A}}^U} = w_{\tilde{\tilde{B}}^U} = 1$. Based on formulas (34) and (38), we can know

$$S(\tilde{\tilde{A}}^\Delta, \tilde{\tilde{B}}^\Delta) = \left[1-\frac{\sqrt{\sum_{i=1}^{4}(\Delta a_i-\Delta b_i)^2}}{2}\right]\times\left[1-\sqrt{\frac{|\Delta S_a-\Delta S_b|}{2}}\right]\times\left[1-\frac{|w_{\tilde{\tilde{A}}^L}-w_{\tilde{\tilde{B}}^L}|}{|w_{\tilde{\tilde{A}}^U}+w_{\tilde{\tilde{B}}^U}|}\right]$$

$$=\left[1-\frac{\sqrt{0}}{2}\right]\times\left[1-\sqrt{\frac{0}{2}}\right]\times\left[1-\frac{|0|}{2}\right]$$

=1.

$$S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U) = \left[1-\frac{\sqrt{\sum_{i=1}^{4}\left(a_i^U-b_i^U\right)^2}}{2}\right]\times\left[1-\sqrt{\frac{|S_{\tilde{\tilde{A}}^U}-S_{\tilde{\tilde{B}}^U}|}{2}}\right]\times\frac{\min(w_{\tilde{\tilde{A}}^U}, w_{\tilde{\tilde{B}}^U})}{\max(w_{\tilde{\tilde{A}}^U}, w_{\tilde{\tilde{B}}^U})}$$

$$=\left[1-\frac{\sqrt{4\times(a_i^U-b_i^U)^2}}{2}\right]\times\left[1-\sqrt{\frac{0}{2}}\right]\times 1$$

$$=\left[1-\frac{2|a-b|}{2}\right]=1-|a-b|.$$

Therefore,

$$S(\tilde{\tilde{A}}, \tilde{\tilde{B}}) = \frac{S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U)\times(1+S(\tilde{\tilde{A}}^\Delta, \tilde{\tilde{B}}^\Delta))}{2}$$

$$= \frac{1-|a-b\times(1+1)|}{2}$$

=1-|a-b|. Q.E.D

Assume that there are two interval-valued trapezoidal fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$,

$\tilde{\tilde{A}}$ =[(0.3,0.35,0.45,0.5; 0.8), (0.1,0.25,0.55,0.7; 1.0)],

$\tilde{\tilde{B}}$ =[(0.25,0.3,0.4,0.45; 0.8), (0.05,0.2,0.5,0.65; 1.0)].

The degree of similarity between the interval-valued trapezoidal fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ is calculated as follows:

[Step 1] Based on formula (33), the distance values Δa_i on the X-axis between the lower and upper trapezoidal fuzzy numbers $\tilde{\tilde{A}}^L$ and $\tilde{\tilde{A}}^U$ of the interval-valued trapezoidal fuzzy number $\tilde{\tilde{A}}$ are calculated as Δa_1=0.2, Δa_2=0.1, Δa_3=0.1, and Δa_4 = 0.2. In the same way, the distance values Δb_1=0.2, Δb_2=0.1, Δb_3 = 0.1, and Δb_4 = 0.2.

[Step 2] Based on formula (34), the degree of similarity $S(\tilde{\tilde{A}}^\Delta, \tilde{\tilde{B}}^\Delta)$ between the distance values Δa_i and Δb_i of the two interval-valued trapezoidal fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ can be calculated as follows:

$$S(\tilde{\tilde{A}}^\Delta, \tilde{\tilde{B}}^\Delta) = \left[1-\frac{\sqrt{\sum_{i=1}^{4}(\Delta a_i-\Delta b_i)^2}}{2}\right]\times\left[1-\sqrt{\frac{|\Delta S_a-\Delta S_b|}{2}}\right]\times\left[1-\frac{|w_{\tilde{\tilde{A}}^L}-w_{\tilde{\tilde{B}}^L}|}{|w_{\tilde{\tilde{A}}^U}+w_{\tilde{\tilde{B}}^U}|}\right]$$

$$=\left[1-\frac{\sqrt{0^2+0^2+0^2+0^2}}{2}\right]\times\left[1-\sqrt{\frac{0.1826-0.1826}{2}}\right]\times\left[1-\frac{|0.8-0.8|}{|1+1|}\right]$$

$$=1\times1\times1$$

$$=1.$$

where Δs_a and Δs_b be calculated by formulas (35),(36) and (37).

[Step 3] Based on formula (38), the degree of similarity $S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U)$ between the upper trapezoidal fuzzy numbers $\tilde{\tilde{A}}^U$ and $\tilde{\tilde{B}}^U$ can be calculated as follows:

$$S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U)=\left[1-\frac{\sqrt{\sum_{i=1}^{4}\left(a_i^U-b_i^U\right)^2}}{2}\right]\times\left[1-\sqrt{\frac{\left|S_{\tilde{\tilde{A}}^U}-S_{\tilde{\tilde{B}}^U}\right|}{2}}\right]\times\frac{\min(w_{\tilde{\tilde{A}}^U},w_{\tilde{\tilde{B}}^U})}{\max(w_{\tilde{\tilde{A}}^U},w_{\tilde{\tilde{B}}^U})}$$

$$=\left[1-\frac{\sqrt{0.05^2+0.05^2+0.05^2+0.05^2}}{2}\right]\times\left[1-\sqrt{\frac{0.2739-0.2739}{2}}\right]\times\frac{1}{1}$$

$$=0.95.$$

where $s_{\tilde{\tilde{A}}^U}$ and $s_{\tilde{\tilde{B}}^U}$ be calculated by formula (36).

[Step 4] Based on formula (39), the degree of similarity $S(\tilde{\tilde{A}}, \tilde{\tilde{B}})$ between the interval-valued trapezoidal fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ can be calculated as follows:

$$S(\tilde{\tilde{A}}, \tilde{\tilde{B}}) = \frac{S(\tilde{\tilde{A}}^U,\tilde{\tilde{B}}^U)\times(1+S(\tilde{\tilde{A}}^\Delta,\tilde{\tilde{B}}^\Delta))}{2}$$

$$=\frac{0.95\times(1+1)}{2}$$

$$=0.95.$$

Therefore, the degree of similarity between the interval-valued trapezoidal fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ is 0.95.

IV. Comparing Existing Methods with the Proposed Similarity Measure

In this section, we compare the proposed similarity measure with five existing similarity measures [6], [8], [12], [21], [10] using 17 sets of interval-valued fuzzy numbers, as illustrated in Figure 2. Some of these sets are derived from prior studies [6], [8], [12], [21], [10]. Generally, the degree of similarity between two interval-valued fuzzy numbers is determined by three factors: the similarity of the shapes and spreads of their membership functions, the relative distance between the two fuzzy numbers, and their alignment in fuzzy number ranking problems [11]. Parts of the 17 sets of generalized fuzzy numbers are sourced from [6], [8], [12], [21], [10], while others are extensions of Sets 16 and 17. These sets are constructed based on the aforementioned criteria.

Table I presents the calculation results for all five similarity measures. Both Table I and Figure 2 reveal that the existing similarity measures [6], [8], [12], [21], [10] exhibit certain limitations, which are described in detail below:

(1) In Set 4 of Figure 2, the two interval-valued fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{C}}$ are more similar than the two interval-valued fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$, because the shapes of the latter two differ more than those of the former two, and the relative distance between the former two is the same as that between the latter two. However, Table I shows that the methods of Chen and Chen [6], Chen [8], and Wei and Chen [21] yield an incorrect result that the two interval-valued fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ are more similar than the two interval-valued fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{C}}$.

(2) In Set 5 of Figure 2, the two interval-valued fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ are more similar than the two interval-valued fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{C}}$, because the shapes of the latter two are more different than those of the former two, and the relative distance between the former two is the same as that between the latter two. However, Table I indicates that applying Chen's method [8] yields the same degree of similarity (i.e., $S(\tilde{\tilde{A}}, \tilde{\tilde{B}}) = S(\tilde{\tilde{A}}, \tilde{\tilde{C}})$ for the two sets $(\tilde{\tilde{A}}, \tilde{\tilde{B}})$ and $(\tilde{\tilde{A}}, \tilde{\tilde{C}})$ of these interval-valued fuzzy numbers $\tilde{\tilde{A}}$, $\tilde{\tilde{B}}$ and $\tilde{\tilde{C}}$.

(3) In Set 6 of Figure 2, the degrees of similarity $S(\tilde{\tilde{A}}, \tilde{\tilde{B}})$ and $S(\tilde{\tilde{A}}, \tilde{\tilde{C}})$ of the two sets of interval-valued fuzzy numbers $(\tilde{\tilde{A}}, \tilde{\tilde{B}})$ and $(\tilde{\tilde{A}}, \tilde{\tilde{C}})$ are different. However, Table I shows that Chen's method [8] yields the same degrees of similarity for the two sets $(\tilde{\tilde{A}}, \tilde{\tilde{B}})$ and $(\tilde{\tilde{A}}, \tilde{\tilde{C}})$ of interval-valued fuzzy numbers $\tilde{\tilde{A}}$, $\tilde{\tilde{B}}$ and $\tilde{\tilde{C}}$.

(4) In Set 7 of Table I, the degree of similarity between the interval-valued fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{C}}$ cannot be correctly calculated using Chen and Chen's Method [12], because the denominator $w_{\tilde{\tilde{C}}^L}$ of formula (7) would become zero, producing the incorrect result $w_{\tilde{\tilde{C}}^L} = \infty$. Furthermore, in Set 7 of Figure 2, the degree of similarity $S(\tilde{\tilde{A}}, \tilde{\tilde{C}})$ is not zero. However, Table I indicates that Chen's method [8] yields an incorrect result $S(\tilde{\tilde{A}}, \tilde{\tilde{C}}) = 0$.

(5) In Set 8 of Table I, the degrees of similarity $S(\tilde{\tilde{A}}, \tilde{\tilde{B}})$ and $S(\tilde{\tilde{A}}, \tilde{\tilde{C}})$ cannot be correctly calculated using [12] because $w_{\tilde{\tilde{A}}^L}$ and $w_{\tilde{\tilde{B}}^L}$ in formula (7) are zero, yielding the incorrect results $x^*_{\tilde{\tilde{A}}^L} = \infty$ and $x^*_{\tilde{\tilde{B}}^L} = \infty$. The $S(\tilde{\tilde{A}}, \tilde{\tilde{B}})$ cannot be correctly calculated using Chen's method [8], because the denominator $\max(y^*_{\tilde{\tilde{A}}^L}, y^*_{\tilde{\tilde{B}}^L})$ in formula (9) becomes zero, producing the incorrect result $S(\tilde{\tilde{A}}^L, \tilde{\tilde{B}}^L) = \infty$. Additionally, in Set 8 of Figure 2, the degree of similarity $S(\tilde{\tilde{A}}, \tilde{\tilde{C}})$ is not zero. However, Table I indicates that the methods of Chen's method [8] yields an incorrect result $S(\tilde{\tilde{A}}, \tilde{\tilde{C}})=0$. The degrees of similarity $S(\tilde{\tilde{A}}, \tilde{\tilde{B}})$ and $S(\tilde{\tilde{A}}, \tilde{\tilde{C}})$ of the two sets of fuzzy numbers $(\tilde{\tilde{A}}, \tilde{\tilde{B}})$ and $(\tilde{\tilde{A}}, \tilde{\tilde{C}})$ are different. However, Table I demonstrates that Chen and Chen's method [12] yields the same degree of similarity for the two sets $(\tilde{\tilde{A}}, \tilde{\tilde{B}})$ and $(\tilde{\tilde{A}}, \tilde{\tilde{C}})$ of fuzzy numbers $\tilde{\tilde{A}}$, $\tilde{\tilde{B}}$ and $\tilde{\tilde{C}}$.

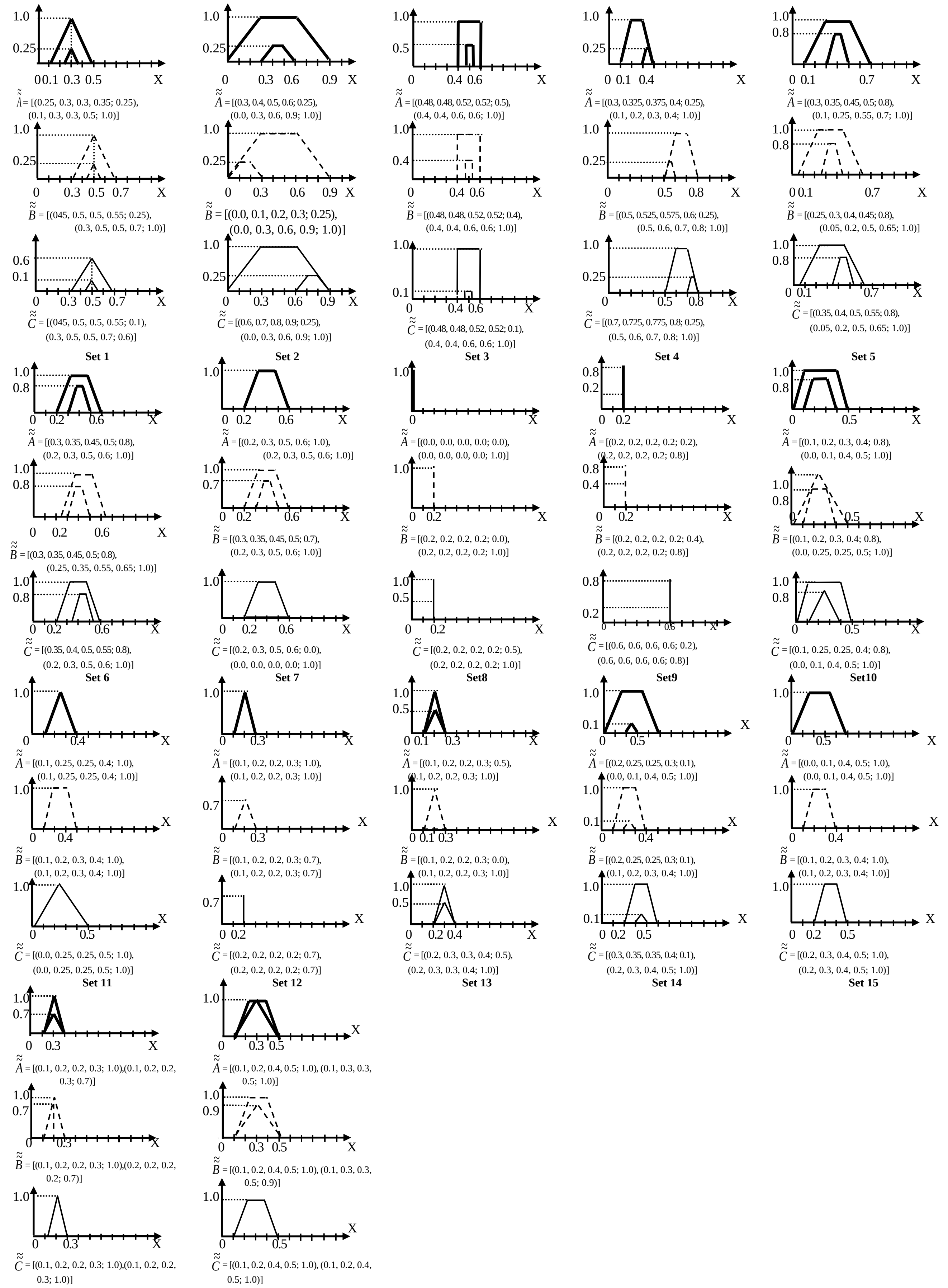

Fig. 2. The 17 sets of interval-valued fuzzy numbers.[10]

TABLE I Comparison of the Calculation Results of the Proposed Similarity Measure and the Existing Methods

	Chen and Chen's Method [6]		Chen's Method [8]		Chen-and-Chen's Method [12]		Wei-and-Chen' s Method [21]		Chen's Method [10]		The Proposed Method	
	$S(\tilde{\tilde{A}},\tilde{\tilde{B}})$	$S(\tilde{\tilde{A}},\tilde{\tilde{C}})$	$S(\tilde{\tilde{A}},\tilde{\tilde{B}})$	$S(\tilde{\tilde{A}},\tilde{\tilde{C}})$	$S(\tilde{\tilde{A}},\tilde{\tilde{B}})$	$S(\tilde{\tilde{A}},\tilde{\tilde{C}})$	$S(\tilde{\tilde{A}},\tilde{\tilde{B}})$	$S(\tilde{\tilde{A}},\tilde{\tilde{C}})$	$S(\tilde{\tilde{A}},\tilde{\tilde{B}})$	$S(\tilde{\tilde{A}},\tilde{\tilde{C}})$	$S(\tilde{\tilde{A}},\tilde{\tilde{B}})$	$S(\tilde{\tilde{A}},\tilde{\tilde{C}})$
Set 1	0.8	0.3647	0.8	0.3919	0.8	0.4115	0.8618	0.681	0.8	0.444	0.798	0.4575
Set 2	0.8367	0.8367	0.8367	0.8367	0.7	0.7	0.9386	0.9386	0.874	0.874	0.8677	0.8677
Set 3	0.8944	0.4472	0.8944	0.4472	0.9983	0.9814	0.9668	0.8475	0.95	0.8	0.975	0.9
Set 4	0.6928	0.4559	0.6928	0.6	0.48	0.6	0.7402	0.7114	0.5621	0.6	0.5563	0.598
Set 5	0.95	0.9372	0.95	0.95	0.95	0.855	0.9664	0.9539	0.95	0.9025	0.95	0.9025
Set 6	0.9747	0.9616	0.9747	0.9747	0.9025	0.95	0.9632	0.9805	0.9263	0.975	0.9261	0.975
Set 7	0.8205	*	0.8046	0	0.4477	0.4167	0.8079	0.5	0.8237	0.5	0.8078	0.75
Set 8	*	*	*	0	0.8	0.8	0.4	0.2828	0.8	0.6	0.8	0.7
Set 9	0.7071	0.6	0.7071	0.6	1	0.6	0.6708	0.6	0.9	0.6	0.9375	0.6
Set 10	0.8601	0.9018	0.8429	0.9141	0.7101	0.9652	0.9283	0.9875	0.9119	0.9874	0.7149	0.9546
Set 11	0.8464	0.9682	0.8356	0.9494	0.9686	0.8783	0.95	0.8862	0.9748	0.9494	0.9091	0.7415
Set 12	0.7	0.9283	0.7	0.9042	0.49	0.4304	0.7209	0.6215	0.595	0.5649	0.6382	0.4733
Set 13	*	0.9	0	0.9	0.8333	0.9	*	0.9322	0.75	0.9	0.875	0.9
Set 14	0.9227	0.8279	0.8987	0.8513	0.7009	0.7009	0.9533	0.9031	0.855	0.8524	0.583	0.5562
Set 15	0.8514	0.7843	0.8077	0.8053	0.8116	0.8116	0.8055	0.8055	0.9	0.8974	0.69	0.6582
Set 16	0.8061	0.8367	0.7956	0.8367	0.7283	0.5667	0.8937	0.8605	0.9747	0.85	0.8708	0.925
Set 17	0.9487	0.9413	0.9487	0.929	0.9667	0.4117	0.9744	0.9764	0.95	0.9747	0.975	0.919

Note: " * " means that the similarity measure cannot calculate the degree of similarity between two interval-valued fuzzy numbers.
" ▒ " means incorrect results

(6) In Set 9 of Figure 2, the interval-valued fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ are not the same, because their shapes are different. However, according to Table I, Chen and Chen's method [12] yields an incorrect result $S(\tilde{\tilde{A}},\tilde{\tilde{B}}) = 1$.

(7) In Set 11 of Figure 2, the two interval-valued fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ have higher similarity than the two interval-valued fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{C}}$, because the shapes of the latter two are more different than those of the former two, and the relative distance between the latter two is the same as that between the former two. However, Table I indicates that the methods of Chen and Chen [6], and Chen [8] yield an incorrect result that the two interval-valued fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{C}}$ are more similar than the two interval-valued fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$.

(8) In Set 12 of Figure 2, the two interval-valued fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ are more similar than the two interval-valued fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{C}}$, because the shapes of the latter two differ more than those of the former two, and the relative distance between the latter two equals that between the former two. However, Table I indicates that the methods of Chen and Chen [6], and Chen [8] yield an incorrect result that the two interval-valued fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{C}}$ are more similar than the two interval-valued fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$.

(9) In Set 13 of Table I, the degree of similarity $S(\tilde{\tilde{A}},\tilde{\tilde{B}})$ cannot be correctly determined using Chen and Chen's method [6] and Wei-and-Chen' s method [21], because the denominators $w_{\tilde{\tilde{A}}^L}$ and $w_{\tilde{\tilde{B}}^L}$ in formula (7) would be zero, producing the incorrect results $x^*_{\tilde{\tilde{A}}^L} = \infty$ and $x^*_{\tilde{\tilde{B}}^L} = \infty$. Furthermore, in Set 13 of Figure 2, the degree of similarity $S(\tilde{\tilde{A}},\tilde{\tilde{B}})$ is not zero. However, Table I indicates that Chen's method [8] yields the incorrect result $S(\tilde{\tilde{A}},\tilde{\tilde{C}}) = 0$.

(10) In Set 14 of Figure 2, the degrees of similarity $S(\tilde{\tilde{A}},\tilde{\tilde{B}})$ and $S(\tilde{\tilde{A}},\tilde{\tilde{C}})$ of the two sets of interval-valued fuzzy numbers $(\tilde{\tilde{A}},\tilde{\tilde{B}})$ and $(\tilde{\tilde{A}},\tilde{\tilde{C}})$ are different. However, Table I indicates that Chen and Chen's method [12] yields the same degrees of similarity for the two sets $(\tilde{\tilde{A}},\tilde{\tilde{B}})$ and $(\tilde{\tilde{A}},\tilde{\tilde{C}})$ of interval-valued fuzzy numbers $\tilde{\tilde{A}}$, $\tilde{\tilde{B}}$ and $\tilde{\tilde{C}}$.

(11) In Set 15 of Figure 2, the degrees of similarity $S(\tilde{\tilde{A}},\tilde{\tilde{B}})$ and $S(\tilde{\tilde{A}},\tilde{\tilde{C}})$ of the two sets of interval-valued fuzzy numbers $(\tilde{\tilde{A}},\tilde{\tilde{B}})$ and $(\tilde{\tilde{A}},\tilde{\tilde{C}})$ are different. However, Table I indicates that the methods of Chen and Chen [12], and Wei-and-Chen [21] yield the same degrees of similarity for the two sets $(\tilde{\tilde{A}},\tilde{\tilde{B}})$ and $(\tilde{\tilde{A}},\tilde{\tilde{C}})$ of interval-valued fuzzy numbers $\tilde{\tilde{A}}$, $\tilde{\tilde{B}}$ and $\tilde{\tilde{C}}$.

(12) In Set 16 of Figure 2, the two interval-valued fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{C}}$ are more similar than the two

interval-valued fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$, because the shapes of the latter two differ more than those of the former two, and the relative distance between the interval-valued fuzzy number $\tilde{\tilde{A}}$ is the same as that between the interval-valued fuzzy number $\tilde{\tilde{C}}$. However, Table I shows that the methods of Chen and Chen [12], and Wei and Chen [21] and Chen [10] yield an incorrect result that the two interval-valued fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ are more similar than the two interval-valued fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{C}}$.

(13) In Set 17 of Figure 2, the two interval-valued fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ are more similar than the two interval-valued fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{C}}$, because the shapes of the latter two differ more than those of the former two, and the relative distance between the interval-valued fuzzy number $\tilde{\tilde{A}}$ is the same as that between the interval-valued fuzzy number $\tilde{\tilde{B}}$.However, Table I shows that the methods of Wei and Chen [21] and Chen [10] yield an incorrect result that the two interval-valued fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{C}}$ are more similar than the two interval-valued fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$.

Table I and Figure 2 clearly indicate that the proposed similarity measure overcomes the drawbacks of the existing methods.

V. Conclusion

This study presents a new approach for calculating similarity measure between interval-valued trapezoidal fuzzy numbers. Some properties of the proposed similarity measure were demonstrated, and 17 sets of generalized fuzzy numbers were adopted to compare the proposed similarity measure with five existing similarity measures. Table I indicate that the proposed similarity measure overcomes the drawbacks of the existing similarity measures. The proposed similarity measure provides a useful way to calculate the degree of similarity between internal-valued trapezoidal fuzzy numbers.

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