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Chebyshev Approximation For COS (½ πX4) & SIN (½ πX4)

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***Abstract*— In this paper , We have extended Cohen's work[1] to approximate cos(½ πx4) and sin(½ πx4) ,**

$0 \le x \le 1$, in terms of Chebyshev polynomials . We have used two approaches to effect the agreement in the result of computations for Chebyshev coefficients to 8 digits of accuracy. The first approach is the fundamental Chebyshev IDENTITY derived by Cohen [1]. However, We have DERIVED this IDENTITY again in THEOREM 1 and We have observed that each entry element in Table 2 DERIVED by Cohen's work [1] and listed in Appendix A can be evaluated separately.

This , in contrast to COHEN'S DERIVATION [1] in which each individual entry in Table 2 can not be evaluated separately unless the evaluations of the previous rows are calculated first . The second method, we have apply is the direct series summation techniques.

***Index Terms*— Chebyshev Identities,Clenshaw**

I. INTRODUCTION

The BACKGROUND of this paper is that Clenshaw [4] has approximated sin (½ πx) & cos (½ πx) , $-1 \le x \le 1$ as an expansion of Chebyshev polynomials . The coefficients are determined exactly in terms of Bessel functions and then approximated numerically to 20 digits of accuracy as in [4]. Cohen [1] has extended Clenshaw's work [4] to approximate cos (½ πx^2) & sin (½ πx^2), $0 \le x \le 1$ as an expansion of Chebyshev series. The coefficients are determined to 20 digits of accuracy using two approaches.

The first approach he has derived a fundamental Chebyshev IDENTITY for expressing Chebyshev polynomials $T_r(x^2)$ as a linear approximation in Chebyshev polynomials Tj(x), r, j=0(1) n. This fundamental IDENTITY that We shall derived again produces Table 2 in Appendix A. The second approach in [1] was the continued fraction technique. One APPLICATION of this, demonstrated, in Cohen's work[1] is the evaluation of FRESNEL INTEGRAL .

In Cohen's work [1], Chebyshev polynomials approximations are expressed as

$$\cos \tfrac{1}{2}\pi x^2 = \sum ß_{2r}\, T_{2r}(x) \qquad (1)$$

and

$$\sin \tfrac{1}{2}\pi x^2 = \sum (-1)^r\, ß_{2r}\, T_{2r}(x) \qquad (2)$$

where the coefficients ß's are given in Table 1. The evaluations of the coefficients ß's in Table 1 are determined by establishing Chebyshev identities for $T_r(x^2)$ as described by Cohen [1].

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That is

$$2^r\, T_r(x^2) = \sum_{k=0}^{r} \alpha_{r,k}\, T_{2r-2k}(x) \qquad (3)$$

where the coefficients α's are listed in Table 2.

We observed that the determination of a single entry in Table 2 is not feasible in Cohen's derivation [1] unless the evaluations for the entries start from the first few rows. We overcome such a difficulty by

Table 1 Coefficients in the expansion for sin½πx² and cos½πx² (0≤ x ≤ 1)

$\cos \tfrac{1}{2}\pi x^2 = \sum ß_{2r}T_{2r}(x)$, $\sin\tfrac{1}{2}\pi x^2 = \sum (-1)^r ß_{2r}T_{2r}(x)$

2r		**ß₂ᵣ**	
0		0.60219 47012 55546 40329	
2	-	0.51362 51666 79107 02511	
4	-	0.10354 63442 62963 75381	
6	+	0.01373 20342 34358 55321	
8	+	0.00135 86698 38090 36178	
10	-	0.00010 72630 94406 00221	**Checks are**
12	-	70462 96793 46857	**provided by**
14	+	3963 90250 61486	**$\sum (-1)^r ß_{2r} = 1$**
16	+	194 99597 75588	**& $\sum ß_{2r} = 0$**
18	-	8 52292 89262	
20	-	33516 50652	
22	+	1197 93739	
24	+	39 24123	
26	-	1 18639	
28	-	3330	
30	+	87	
32	+	2	

deriving a formula for Table 2 again and thus each entry element in it can be evaluated directly.

II. CHEBYSHEV IDENTITIES

The first well known identity that we have rewritten , is the explicit representation of x^n in terms of Chebyshev polynomial $T_j(x)$, j=0(1)n , for some positive integer n.

That is

$$x^n = {\sum_{j=0}^{n}}' c_j\, T_j(x) \qquad (4)$$

where $\quad c_j = 2^{-n+1} C^{n}_{(j+n)/2} \qquad (5)$

for j =1(1)n if n is odd & j =0(2)n if n is even . The prime dash in the summation means that coefficient of $T_0(x)$ in (4) should be halved .

The next identity that we have derived again without proof is the expansion of Chebyshev polynomials $T_n(x)$ as successive powers in x , where n is a positive integer $T_n(x)$

$$= \sum_{j=0}^{r} \alpha_j x^{n-2j} \qquad (6)$$

where the coefficients α's are determined by

$$\alpha_0 = 2^{n-1}$$

and $$\alpha_j = (-1)^j 2^{n-2j-1} C_{j-1}^{n-j-1} \frac{n}{j} \qquad (7)$$

for j = 1 (1) r and r = [n ⁄ 2] is the integer part of n ⁄ 2 Where C_{j-1}^{n-j-1} is the Corresponding Binomial coefficient. For an alternative expression of Eqn. (6), see [2].

The third identity that we shall derive again, is the expansion of $T_n(x^2)$ in terms of Chebyshev series, as shown in Theorem 1. For an alternative derivation, see [1].

Theorem 1.

If $T_n(x)$ is the Chebyshev polynomial of degree n, where n is a positive integer , then we can expand $T_n(x^2)$ in terms of a Chebyshev series as

$$T_n(x^2) = \sum_{j=0}^{n}{}' a_{n,k} T_{2k}(x) \qquad (8)$$

where

$$a_{n,k} = \sum_{j=0}^{r} \alpha_j 2^{4j-2n+1} C_{n-2j+k}^{2n-4j} \qquad (9)$$

for k=0(1)n , r = [n ⁄ 2] and α_j for j = 0(1)r are described by Eqn .(7). The prime dash on the sign sigma means the coefficients $a_{n,0}$ are to be halved , for each integer n≥1

Proof

Eqn.(6) implies that

$$T_n(x^2) = \sum_{j=0}^{r} \alpha_j x^{2n-4j} \qquad (10)$$

while Eqn.(5) implies

$$x^{2n-4j} = \sum_{k=0}^{n-2j} c_k T_{2k}(x \qquad (11)$$

hence , We can write

$$T_n(x^2) = \sum_{k=0}^{n} a_{n,k} T_{2k}(x)$$

with

$$a_{n,k} = \frac{2}{\pi} \int_{-1}^{1} \frac{T(x^2) T_{2k}(x)}{\sqrt{1-x^2}} dx \quad , \quad k = 0(1)n \qquad (12)$$

now

$$a_{n,k} = \sum_{j=0}^{r} \alpha_j \left(\frac{2}{\pi} \int_{-1}^{1} \frac{x^{2n-4j} T_{2k}(x)}{\sqrt{1-x^2}} dx \right) \quad (13)$$

The method describes the evaluations of the integral enclosed within the parentheses can be found in Froberg's book[3] , page(79).

Hence , We get

$$a_{n,k} = \alpha_j 2^{4j-2n+1} C_{n-2j+k}^{2n-4j} \qquad (14)$$

for k=0(1)n , where

α_j for j=0(1)n are given by Eqn .(7) and r = [n ⁄ 2].

We remark that the elements $a_{n,k}$, for k =0(1)n , are the entries for Table 2 derived earlier by Cohen's work[1] . If initial coefficients $a_{n,0}$ are halved , then , we can compute each entry element that lies anywhere inside Table 2 . For example , further elements in Table 2 are given by $a_{25,23} = 1125$ and $a_{29,27} = 1537$

III. APPROXIMATION OF COS ½πX⁴ AND SIN ½πX⁴

We shall first apply chebyshev expansion series for cos $½πx^4$, where the general coefficient is evaluated using direct series summation technique . That is when

$$\cos ½\pi x^4 = \sum_{r=0}^{n}{}' ß_r T_{2r}(x) \qquad (15)$$

then $$ß_r = \frac{2}{\pi} \int_{-1}^{1} \frac{\cos ½ \pi x^4 T_{2k}(x)}{\sqrt{1-x^2}} dx \qquad (16)$$

If We expand $\cos ½πx^4$ as

$$\cos ½\pi x^4 = \sum_{j=0}^{\infty} (-1)^j (½\pi)^{2j} x^{8j} ⁄ (2j!) \qquad (17)$$

and substitute x = cosθ , then Eqn. (16) can be rewritten as

$$ß_r = \sum_{j=0}^{\infty} \frac{(-1)^j (½\pi)^{2j}}{(2j!)} *$$

$$* \left(\frac{2}{\pi} \int_0^{\pi} (\cos\theta)^{8j} \cos 2r\theta \, d\theta \right) \qquad (18)$$

hence

$$ß_r = \sum_{j=0}^{\infty} (-1)^j (½\pi)^{2j} 2^{-8j+1} C_{4j+r}^{8j} ⁄ (2j!) \qquad (19)$$

for r = 0(1)n

We observe that the terms in the series (19) are decreasing rapidly to zero and all the terms are identically zero whenever r >4j.

Alternatively, We knew that if we replace x by x^2 in Eqn. (1), We get

$$\cos ½\pi x^4 = \sum ß_{2r} T_{2r}(x^2) \qquad (20)$$

By using the values of $ß_{2r}$, r = 0(1)16 , given in Table 1 and Chebyshev identities for $T_r(x^2)$, We get

$$\cos ½\pi x^4 = \sum ß_r T_{2r}(x) \qquad (21)$$

where the coefficients ß's in Eqn.(21) are exactly given by Eqn.(19) . The values of the coefficients ß's are shown in Table 3 rounded to 8 significant digits according to the agreement of the evaluation by both methods.

Similarly, the approximation for sin $½πx^4$ can be evaluated by direct series summation or Chebyshev identities for $T_r(x^2)$ as

$$\sin ½\pi x^4 = \sum ß_r T_{2r}(x) \qquad (22)$$

where

$$ß_r = \sum_{j=0}^{\infty} (-1)^j (½\pi)^{2j+1} 2^{-8j-3} C_{4j+r+2}^{8j+4} /(2j+1)! \quad (23)$$

for r = 0 (1)n

The values for the coefficients ß's given by Eqn.(23) are listed in Table 4 rounded to 8 significant digits according to the agreement in the results by both methods

Table 3 The coefficients ß's in the expansion for cos $½\pi x^4$

cos ½πx4 = ∑ ßrT2r(x)

r	ßr	
0	0.70923853	
1	-0.45715459	
2	-0.21256022	
3	-0.04639916	
4	+0.00271565	
5	+0.00352780	**Checks are**
6	+0.00062078	**provided by**
7	+0.00002962	**∑(-1)r ßr = 1**
8	-0.00001443	
9	-0.00000370	
10	-0.00000034	
11	-0.00000001	

Table 4 The coefficients ß's in the expansion for sin ½πx4

sin ½πx4 = ∑ ßrT2r(x)

r	ßr	
0	0.45669090	
1	+0.55985521	
2	+0.05834041	
3	-0.05833114	
4	-0.01536607	**Checks are**
5	-0.00165872	**provided by**
6	+0.00031830	**∑(-1)r ßr = 0**
7	+0.00013469	
8	+0.00001688	
9	+0.00000003	
10	-0.00000043	
11	-0.00000008	

Appendix A

Table 2

Coefficients $\alpha_{r-k}^{(r)}$ **in expansion of** $2^r T_r(x^2)$

r \ k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	1																
1	1	1															
2	-1	4	1														
3	-2	3	6	1													
4	3	-8	12	8	1												
5	6	-10	0	25	10	1											
6	-10	24	-33	28	42	12	1										
7	-20	35	-14	-35	84	63	14	1									
8	35	-80	104	-112	28	176	88	16	1								
9	70	-126	72	48	-216	216	312	117	18	1							
10	-126	280	-350	400	-280	-176	605	500	150	20	1						
11	-252	462	-308	-22	528	-891	286	1287	748	187	22	1					
12	462	-1008	1224	-1424	1263	-264	-1452	1608	2370	1064	228	24	1				
13	924	-1716	1248	-273	-1274	2847	-2704	-1014	4420	3978	1456	273	26	1			
14	-1716	3696	-4389	5124	-5026	2772	2233	-6744	2324	9576	6251	1932	322	28	1		
15	-3432	6435	-4950	1925	2940	-8637	11310	-4875	-10200	11815	18186	9345	2500	375	30	1	
16	6435	-13728	16016	-18656	19272	-14112	176	18336	-22756	-6688	32528	31648	13432	3168	432	32	1

IV. COMPUTATIONAL REMARKS

The accumulations of rounding errors due to the limitedness machine's accuracy of I.B.M. Micro computers applied with Basic Language , effect the evaluations of the coefficients in Table 3 &Table 4 to be taken to 8 significant digits .

However , both the methods described can retain more accuracy for the evaluations of the coefficients in Table 3 & 4 , whenever recomputation is taken on sufficient machine's accuracy .The continued fraction technique can be used instead of direct series technique .

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