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Optimal Control Analysis of the Mathematical Model for HIV Infection of $CD4^+$ T Cells with Treatment using Adomian Decomposition Method

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***Abstract*— This study investigated Human Immunodeficiency Virus (HIV) models of Liancheng and Micheal. The model was extended by incorporating the treatment term ρ and a polynomial of the form $\frac{1}{1+\alpha_1 v}$ which gives information about the current and past states of the virus. The optimality system was derived and an optimal control model of drug treatment of HIV infection of $CD4^+$T-cells was investigated. Conditions for the optimal control were considered using Pontryagin's maximum principle and solve numerically using Adomian Decomposition Method (ADM). Global stability of the equilibria, the existence and uniqueness of the solution to the problem for the optimal control pair were established. The model exhibit two equilibria, disease-free and endemic equilibrium. The simulated optimal control pair (u_1, u_2) controls the percentage effect of the chemotherapy on the $CD4^+$T-cells and represents the efficiency of the drug treatment in inhibiting viral production and preventing new infections. The characterized objective function based on maximizing T-cells and minimizing the cost of chemotherapy treatment was in agreement with the existing literature. The numerical results obtained using ADM was also in complete agreement DTM. The result obtained also show that the information on the current and past states of the virus would minimize the endemicity of the virus.**

***Index Terms*— HIV infection, Stability, Basic reproduction number, Lyapunov functions, Adomian decomposition method**

I. INTRODUCTION

AIDS is one of the deadliest epidemics in human history. It was first identified in 1981 among homosexual men and intravenous drug users in New York and California. Shortly after its detection in the United States, evidence of AIDS epidemics grew among heterosexual men, women, and children in sub-Saharan Africa. AIDS quickly developed into a worldwide epidemic, affecting virtually every nation. The United Nations Program on HIV/AIDS (UNAIDS) estimates that the worldwide number of new cases of HIV infection peaked in the later 1990s with more than 3 million people nearly infected each year. However, some regions of the world, especially Vietnam, Indonesia and other countries in Southeast Asia, continued to see an increase in the early2000s. In addition, the number of people living with HIV or AIDS has continued to rise as a result of new drug treatments that lengthen life.

AIDS is the final stage of a chronic infection with the human immunodeficiency virus. There are two types of this virus: HIV-1, which is the primary cause of AIDS worldwide, and HIV-2, found mostly in West Africa. Inside the body HIV enters cells of the immune system, especially white blood cells known as T cells. These cells orchestrate a wide variety of disease-fighting mechanisms. A particularly vulnerable to HIV attack are specialized "helper" T cells known as CD_4^+ T-cells. When HIV infects a CD_4^+ T-cells, it commandeers the genetic tools within the cell to manufacture new HIV virus. The newly formed HIV virus then leaves the cell, destroying the CD_4^+ T-cells in the process.

No existing medical treatment can completely eradicate HIV from the body once it has infected human cells. The loss of CD_4^+ T-cells endangers health because these cells help other types of immune cells respond to invading organisms. The average healthy person has over 1,000 CD_4^+ T-cells per micro liter of blood. In a person infected with HIV, the virus steadily destroys CD_4^+ T-cells over a period of years, diminishing the cells' protective ability and weakening the immune system. When the density of CD_4^+ T-cells drops to 200 cells per micro liter of blood, the infected person becomes vulnerable to AIDS-related opportunistic infectious and rare cancers, which take advantage of the weakened immune defenses to cause disease [9]. A model for the interaction of HIV with CD_4^+ T cells that considers four populations: uninfected T cells, latently infected T cells, actively infected T cells and free virus [6]. Effect of AZT on viral growth and T-cell population dynamics were considered and numerical bifurcation techniques were used to map out the parameter regimes of these various behaviours and they showed that when the endemic state is stable, it is characterized by a reduced number of T cells compared with the uninfected state, thus T-cell depletion occurs through the establishment of a new steady state.

An epidemic model of HIV infection of CD_4^+ T-cells with cure rate and delay were examined [8].The dynamics showed that is completely determined by the basic reproduction number $R_0<1$. If $R_0<1$, the disease-free equilibrium is asymptotically stable and the disease dies out. If $R_0>1$, a unique endemic equilibrium exists and is globally stable in the interior of the feasible region. Moreover, they proved the effect of that delay on the stability of the equilibria and

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showed that the introduction of a time delay in the virus-to-healthy cells transmission term can destabilize the system, and periodic solutions can arise through Hopf bifurcation. Model of cell-free viral spread of human immunodeficiency virus (HIV) in a well-mixed compartment such as the blood stream were studied and discussed the existence, stability of the infected steady state and introduced a discrete time delay to a CD_4^+ T-cell and the emission of viral particles on a cellular level [12] .They studied the effect of the time delay on the stability of the endemically infected equilibrium; criteria were given to ensure that the infected equilibrium is asymptotically stable for all delay. Mathematical models were considered for the infection of human immunodeficiency virus-type 1 (HIV-1) with target cells between initial infection, assumed that the infection among the cells can be approximated and the classical mathematical model with nonlinear infection rate [36].

They proved, if $R_0 \le 1$, then the HIV infection is cleared from the T-cells population; if $R_0 > 1$, then the HIV infection persists. Discrete time delay to the model to describe the time between infection of a CD_4^+ T cells and the emission of viral particles on a cellular level were studied [40]. They examined the effect of the time delay on the stability of endemically infected equilibrium; criteria are given to ensure that the infected equilibrium is asymptotically stable for all delay. Also they observed that time delay does not induce instability and oscillations in the model. Mathematical model based upon the assumption that actively—infected helper T-cells come only from the latently-infected T-cell population [37] and they studied the time delay that appears in the equations is assumed to be the average time that a T-cell remains latently infected.

An optimal control model of drug treatment of HIV infection of CD_4^+ T cells were considered [42, 43] .They showed that the optimal controls represent the efficiency of drug treatment in inhibiting viral production and preventing new infections. Existence for the optimal control pair is established and the Pontryagin's maximum principle is used to characterize these optimal controls. The optimality system was derived and solved numerically. The results also showed that the optimal treatment strategies reduce the viral load and increase the uninfected CD_4^+ T cells count, which improves the quality of life of the patient. Series technique for optimal control of HIV infection dynamics was reported [41]. They obtained the minimum amount of medicine required and showed that the number of uninfected CD_4^+ T cells is maximized. Hence, minimum amount of medicine ensured that the effects of the chemotherapy are minimized.

II. Model Formulations

Considered a three dimensional model which consists of concentration of susceptible CD4$^+$T cells (T), concentration of infected CD4$^+$T cells (T*) and free HIV viruses with a general non-linear incidence rate.

Where

$$\frac{dT}{dt} = s - \alpha T + rT\left(1 - \frac{T+T^*}{T_{\max}}\right) - \frac{kVT}{1+\alpha_1 V} + \rho T^* \quad (2.1)$$

$$\frac{dT^*}{dt} = \frac{kVT}{1+\alpha_1 V} - \beta T^* - \rho T^* \quad (2.2)$$

$$\frac{dV}{dt} = N\beta T^* - \gamma V \quad (2.3)$$

Where T, T^* and V denote the size of compartmental of concentration of the uninfected CD_4^+T Cells, infected CD_4^+T and free virus respectively. s is the rate of supply of CD4$^+$T cells from precursors in the bone marrow and thymus, α is the death rate of uninfected CD4$^+$T cells, k is the rate CD4$^+$T cells becomes infected by free virus while T_{max} is the maximum CD4$^+$T cells population level. β is the death rate of infected CD4$^+$T cells, N is the number of free virus produced by lysing a CD4$^+$T cells. γ is the death rate of free virus. r is the rate of growth for the CD4$^+$T cells population. Qualitative investigation of the system describes by Equations (2.1-2.3) reveals that the long-term behaviou, falls into two categories: endemic or dies out. When the diseases dies out naturally, the solution asymptotically approaches a diseases free equilibrium ε_0 of the form,

$$\varepsilon_0 = \left(\frac{(r-\alpha)T_{\max} + \sqrt{T_{\max}\left[(r-\alpha)^2 T_{\max} + 4rs\right]}}{2r}, 0, 0\right)$$

The threshold that determines the stability of this equilibrium is the R_0,

$$R_0 = \frac{kT_0 N\beta}{(\beta+\rho)\gamma}$$

When the disease free equilibrium is unstable, there exists an endemic equilibrium of the form ε_1 is given as

$$\varepsilon_1 = \left(\hat{T}, \hat{T}^*, V\right) \quad (2.4)$$

such that

$$\hat{T} = \frac{\hat{T}^*(\gamma + \alpha_1 N\beta\hat{T}^*)(\beta+\rho)}{kN\beta}$$

$$\hat{T}^* = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$V = \frac{N\beta\hat{T}^*}{\gamma}$$

$$a = \left(\frac{\beta\alpha_1^2}{k^2} + \frac{2\beta\alpha_1^2\rho}{k^2} + \frac{\alpha_1^2\rho^2}{k^2} + \frac{r\alpha_1\beta}{T_{\max}k} + \frac{r\alpha_1^2\rho}{T_{\max}k} \right)$$

$$b = \frac{r}{T_{\max}}\left(\left[\frac{\gamma^2}{k^2N^2} + \frac{2\gamma^2\rho}{k^2N^2\beta} + \frac{\gamma^2\rho}{k^2N^2\beta^2} \right] - \frac{r}{T_{\max}}\left(\frac{2\beta\gamma\alpha_1}{k^2N} + \frac{4\gamma\alpha_1\rho}{k^2N} + \frac{2\gamma\alpha_1\rho^2}{k^2N\beta} \right) \right.$$

$$\left. + \frac{\alpha\alpha_1\beta}{k} + \frac{\alpha\alpha_1\rho}{k} + \frac{r\alpha_1\beta}{k} + \frac{r\alpha_1\rho}{k} + \frac{r\gamma}{kT_{\max}kN} + \frac{r\gamma\rho}{kN\beta T_{\max}} - \beta \right)$$

$$c = s - \frac{\alpha\gamma}{kN} - \frac{\alpha\gamma\rho}{kN\beta} - \frac{r\gamma}{kN} + \frac{r\gamma\rho}{kN\beta}$$

Obviously Equation (2.4) will only exist provided $R_0 > 1$

III. POSITIVITY OF SOLUTIONS

Theorem 3.1: Let the initial data $T(0) > 0, T^*(0) > 0, V(0) > 0,$ then the solutions $T(t), T^*(t),$ $V(t),$ of HIV free model (2.1) are positive for all $t \geq 0$ [45]

Proof: It is clear from the first equation of model (2.1) that

$$\frac{dT}{dt} \geq -\left(\alpha + \frac{rT^*}{T_{\max}} + \frac{kV}{1+\alpha_1 V} \right) T \tag{3.1}$$

$$\frac{dT}{T} \geq -\left(\alpha + \frac{rT^*}{T_{\max}} + \frac{kV}{1+\alpha_1 V} \right) dt \tag{3.2}$$

Integrating both sides of (3.2)

$$InT \geq -\alpha t - \int_0^t \left[\frac{r}{T_{\max}} T^*(\tau) - kV(\tau) \right] d\tau + C_1 \tag{3.3}$$

Take exponential of both sides (3.3) gives

$$T(t) \geq C_2 e^{-\alpha t - \int_0^t \left[\frac{r}{T_{\max}} T^*(\tau) - \frac{kV(\tau)}{1+\alpha_1 V(\tau)} \right] d\tau} \tag{3.4}$$

Applying the initial condition

Hence

$$T(t) \geq T(0) e^{-\alpha t - \int_0^t \left[\frac{r}{T_{\max}} T^*(\tau) - \frac{kV(\tau)}{1+\alpha_1 V(\tau)} \right] d\tau} \tag{3.5}$$

For all $t > 0$

Proof: It is clear from the second equation of model (2.2) that

$$\frac{dT^*}{dt} \geq (\beta + \rho) T^* \tag{3.6}$$

So that,

$$T^*(t) \geq T^*(0) \exp\left[-\int_0^t (\beta + \rho) dz \right] > 0 \tag{3.7}$$

For all $t > 0$

Proof: It is clear from the third equation of model (2.3) that

$$\frac{dV}{dt} \geq (\beta + \gamma) V \tag{3.8}$$

So that,

$$V(t) \geq V(0) \exp\left[-\int_0^t (\beta + \gamma) dz \right] > 0 \tag{3.9}$$

For all $t > 0$

4.1 Global Stability of Virus Free

We shall prove the global stability of the virus-free equilibrium by means of Lyapunov function of the system of Equation (2.1-2.3)

Theorem 4.1: The virus-free equilibrium is globally asymptotically stable if $R_0 < 1$.

Proof: Let us consider the Lyapunov function

$$\overset{*}{L} = \dot{T}^* + \frac{\beta}{N\beta}\dot{V} \tag{4.1}$$

$$= \frac{kVT}{1+\alpha_1 V} - \beta T^* - \rho T^* + \frac{\beta}{N\beta}(N\beta T^* - \gamma V) \tag{4.2}$$

$$= \frac{kVT}{1+\alpha_1 V} - (T^* + \rho)I^* + \frac{\beta^2 N T^*}{N\beta} - \frac{\beta\gamma V}{N\beta} \tag{4.3}$$

$$= \frac{kVT}{1+\alpha_1 V} - \frac{\beta\gamma V}{N\beta} - (\beta + \rho)I^* - \beta T^* \tag{4.4}$$

$$= \left(\frac{\beta\gamma(\beta+\gamma)}{N\beta(1+\alpha_1 V)}\left(\frac{N\beta kVT}{\beta\gamma(\beta+\gamma)} - \frac{1+\alpha_1 V}{\beta+\rho}V\right) - \frac{N\beta(1+\alpha_1 V)}{\beta\gamma}T^*\right) - \beta T^* \tag{4.5}$$

$$= \left(\frac{\beta\gamma(\beta+\gamma)}{N\beta(1+\alpha_1 V)}\left(\frac{N\beta k}{\beta\gamma(\beta+\gamma)}T - \frac{1+\alpha_1}{\beta+\rho}\right)V - \frac{N\beta(1+\alpha_1 V)}{\beta\gamma}T^*\right) - \beta T^* \tag{4.6}$$

$$\leq \left(\frac{\beta\gamma(\beta+\gamma)}{N\beta(1+\alpha_1 V)}\left(\frac{1}{\beta}R_0 - \frac{1+\alpha_1}{\beta+\rho}\right)V - \frac{N\beta(1+\alpha_1 V)}{\beta\gamma}T^*\right) \tag{4.7}$$

$$\leq 0 \tag{4.8}$$

If $\overset{*}{L}$ =0 then $\left(\frac{\beta\gamma(\beta+\gamma)}{N\beta(1+\alpha_1 V)}\left(\frac{N\beta k}{\beta\gamma(\beta+\gamma)}T - \frac{1+\alpha_1}{\beta+\rho}\right)V - \frac{N\beta(1+\alpha_1 V)}{\beta\gamma}T^*\right) = 0$ and $\beta T^* = 0$

Hence, the largest invariant set included in $\{L = 0\}$ is reduced to the virus free equilibrium. Thus by Laselle's invariance principle, the VFE is globally asymptotically stable. [44, 46, 47]

4.2 Global Stability of Endemics Equilibrium

We shall prove the global stability of endemic equilibrium obtained by means of Lyapunov's direct method. The Lyapunov function constructed of the suitable combinations of composite quadratic and logarithmic functions

Theorem 4.2 If R_0>1 then the unique endemic equilibrium ε_1 of equation (2.1-2.3) globally asymptotically stable in the interior of Ω.

Proof: Define $L: \{(T, T^*, V) \in \Omega : T, T^*, V > 0\} \to \Re$ by

$$L(T, T^*, V) = \frac{1}{2}\left[\left(T - \dot{T}\right) + \left(T^* - \dot{T}^*\right) + \left(V - \dot{V}\right)\right]^2 + \left(\frac{\beta + 2\alpha}{k}\right)\left(T^* - \dot{T}^* - \dot{T}^* \ln\frac{T^*}{\dot{T}^*}\right)$$

$$+ \left(\frac{\beta + 2\alpha}{k}\right)\left(V - \dot{V}\right)^2 \tag{4.9}$$

Then L is C^1. on the interior of Ω, ε_1 is the global minimum of L on Ω and $L(\dot{T},\dot{T}^*,\dot{V})=0.$ The time derivative of L computed along solution of equation (2.1-2.32) is

$$L^1=\left[\left(T-\dot{T}\right)+\left(T^*-\dot{T}^*\right)+\left(V-\dot{V}\right)\right]\left(\frac{d(T+T^*+V)}{dt}\right)+\left(\frac{\beta+2\alpha}{k}\right)\left(\frac{T^*-\dot{T}^*}{T^*}\right)\frac{dT^*}{dt}+$$

$$+\left(\frac{\beta+2\alpha}{k}\right)\left(\frac{V-\dot{V}}{V}\right)\frac{dV}{dt} \qquad (4.10)$$

$$=\left[\left(T-\dot{T}\right)+\left(T^*-\dot{T}^*\right)+\left(V-\dot{V}\right)\right]\left(s-\alpha T+rT\left(1-\frac{T+T^*}{T_{\max}}\right)-(1-N)\beta T^*-\gamma V\right)+$$

$$\left(\frac{\beta+2\alpha}{k}\right)\left(\frac{T^*-\dot{T}^*}{T^*}\right)\left(kVT-\beta T^*-\rho T^*\right)+\left(\frac{\beta+2\alpha}{k}\right)\left(\frac{V-\dot{V}}{v}\right)\left(N\beta T^*-\gamma V\right) \qquad (4.11)$$

Using

$$s=\alpha\dot{T}+r\dot{T}\left(1-\frac{\dot{T}+\dot{T}^*}{T_{\max}}\right)+(1-N)\beta\dot{T}^*-\gamma\dot{V}$$

$$kVT-(\beta+\rho)T^*=0$$

$$N\beta T^*-\gamma V=0$$

To write this, we have

$$L'=\left[\left(T-\dot{T}\right)+\left(T^*-\dot{T}^*\right)+\left(V-\dot{V}\right)\right]\left\{-(\alpha+r)\left[\left(T-\dot{T}\right)\left(1-\frac{\dot{T}+\dot{T}^*}{T_{\max}}\right)+\left(V-\dot{V}\right)\right]+(\alpha+r)\left(T^*-\dot{T}^*\right)\right\}$$

$$+(\beta+2\alpha)\left(\left(T-\dot{T}\right)\left(1-\frac{\dot{T}+\dot{T}^*}{T_{\max}}\right)+\left(T^*-\dot{T}^*\right)\right)+\frac{\beta+2\alpha}{k}\left(V-\dot{V}\right)-\left(r+(1-N)\right)-\gamma\left(V-\dot{V}\right) \qquad (4.12)$$

$$L'=(\alpha+r)\left[\left(T-\dot{T}\right)\left(1-\frac{\dot{T}+\dot{T}^*}{T_{\max}}\right)+\left(V-\dot{V}\right)\right]^2-(r+(1-N))\beta\left(T^*-\dot{T}^*\right)^2-\frac{\beta+2\alpha}{k}\left(V-\dot{V}\right)^2 \qquad (4.13)$$

Hence L^1 is negative. Note that, $L^1=0$ if and only if $T=\dot{T},\ T^*=\dot{T}^*and\quad V=\dot{V}$
Therefore the largest compact invariant set in $\left\{(T,T^*,v)\in\Omega:L^1=0\right\}$ is the singleton $\{\varepsilon_1\}$, where ε_1 is the endemic equilibrium. LaSalle's invariant principle then implies that ε_1^* is globally asymptotically stable in the interior of Ω. [44, 46, 47]

5.1 Optimal Control Formulation

We introduced optimal control to the equation (2.1-2.3) then we have new system of the equation below:

$$\frac{dT}{dt} = s - \alpha T + rT\left(1 - \frac{T+T^*}{T_{\max}}\right) - e^{-u_1}\frac{kVT}{1+\alpha_1 V} + \rho T^* \qquad T(t_0) = T_0 \qquad (5.1)$$

$$\frac{dT^*}{dt} = e^{-u_1}\frac{kVT}{1+\alpha_1 V} - \beta T^* - \rho T^* \qquad T^*(t_0) = T_0^* \qquad (5.2)$$

$$\frac{dV}{dt} = e^{-u_2} N\beta T^* - \gamma V \qquad V(t_0) = V_0 \qquad (5.3)$$

The model describes the dynamic interactions among healthy CD_4^+Tcells, infected cells and free virus in the organisms and it is given by Equation (5.1-5.3) above. The influence of the antiretroviral drugs in the model Equation (5.1-5.3) can be simplified by applying. Taylor series decomposition as follows:

$$e^{-u} = 1 - u + \text{higher order term.} \qquad (5.4)$$

Then we obtain:

$$\frac{dT}{dt} = s - \alpha T + rT\left(1 - \frac{T+T^*}{T_{\max}}\right) - (1-u_1)\frac{kVT}{1+\alpha_1 V} + \rho T^* \qquad T(t_0) = T_0 \qquad (5.5)$$

$$\frac{dT^*}{dt} = (1-u_1)\frac{kVT}{1+\alpha_1 V} - \beta T^* - \rho T^* \qquad T^*(t_0) = T_0^* \qquad (5.6)$$

$$\frac{dV}{dt} = (1-u_2)N\beta T^* - \gamma V \qquad V(t_0) = V_0 \qquad (5.7)$$

$$\frac{dW}{dt} = u_2 N\beta T^* - \gamma W \qquad W(t_0) = w_0 \qquad (5.8)$$

5.2 Optimality system

Pontryagin's Maximum principle provides necessary conditions for an optimal control problem. This principle converts problem (5.5-5.8) into a problem of maximizing and Hamiltonian, H, point wisely with respect to u_1 and u_2:

$$H = T(t) - \frac{1}{2}\left[B_1 u_1^2 + B_2 u_2^2\right] + \sum_{i=1}^{4} d_i f_i \qquad (5.9)$$

Where f_i is the right hand side of the differential equation of *i-th* state variable. We define the Hamiltoniann function given by,

$$H(-) = T(t) - \frac{B_1}{2}u_1^2 - \frac{B_2}{2}u_2^2 + \lambda_1\left(s - \alpha T + rT\left(1 - \frac{T+T^*}{T_{\max}}\right) - (1-u_1)\frac{kVT}{1+\alpha_1 V} + \rho T^*\right) +$$

$$\lambda_2\left((1-u_1)\frac{kVT}{1+\alpha_1 V} - \beta T^* - \rho T^*\right) + \lambda_3\left((1-u_2)N\beta T^* - \gamma V\right) + \lambda_4\left(u_2 N\beta T^* - \gamma W\right) \qquad (5.11)$$

Solving the first order condition

$$\frac{\partial H}{\partial u_i} = 0 \qquad \text{and} \qquad \frac{\partial H}{\partial u_2} = 0 \qquad (5.12)$$

Then we obtain

$$\frac{\partial H}{\partial u_i} = -B_1 u_1 + (\lambda_1 - \lambda_2)\frac{kVT}{1+\alpha_1 V} = 0 \qquad (5.13)$$

$$\frac{\partial H}{\partial u_2} = -B_2 u_2 + (\lambda_4 - \lambda_3)N\beta T^* = 0 \qquad (5.14)$$

Then the optimality conditions from (5.12 and 5.13) we obtained;

$$\left.\begin{aligned} u_1 &= \frac{kVT}{(1+\alpha_1 V)B_1}(\lambda_4-\lambda_3) \\ u_2 &= \frac{N\beta T^*}{B_2}(\lambda_4-\lambda_3) \end{aligned}\right\} \tag{5.15}$$

We obtain optimal expressions u_1^* *and* u_2^* to finally we have,

$$u_1^* = \begin{cases} 0 & \frac{kVT}{(1+\alpha_1 V)B_1}(\lambda_1-\lambda_2) \le 0 \\ \frac{kVT}{B_1}(\lambda_4-\lambda_3) & 0 < \frac{kVT}{(1+\alpha_1 V)B_1}(\lambda_1-\lambda_2) < 1 \\ 1 & \frac{kVT}{(1+\alpha_1 V)B_1}(\lambda_1-\lambda_2) \ge 0 \end{cases} \tag{5.16}$$

$$u_2^* = \begin{cases} 0 & \frac{N\beta T^*}{B_2}(\lambda_4-\lambda_3) \\ \frac{N\beta T^*}{B_2}(\lambda_4-\lambda_3) & 0 < \frac{N\beta T^*}{B_2}(\lambda_4-\lambda_3) < 1 \\ 1 & \frac{N\beta T^*}{B_2}(\lambda_4-\lambda_3) \ge 0 \end{cases} \tag{5.17}$$

However, the optimal conditions is given by

$$\left.\begin{aligned} u_1^* &= \min\left(1, \max\left(0, \frac{kVT}{(1+\alpha_1 V)B_1}(\lambda_1-\lambda_2)\right)\right) \\ u_2^* &= \min\left(1, \max\left(0, \frac{N\beta T^*}{B_2}(\lambda_4-\lambda_3)\right)\right) \end{aligned}\right\} \tag{5.18}$$

Utilizing equation (5.18) in (5.5-5.8) we have the following optimality system

$$\frac{dT}{dt} = s - \alpha T + rT\left(1-\frac{T+T^*}{T_{\max}}\right) - \left(1-\min\left\{1, \max\left\{0, \frac{kVT}{(1+\alpha_1 V)B_1}(\lambda_1-\lambda_2)\right\}\right\}\right)\frac{kvT}{(1+\alpha_1 V)}$$

$$+\rho T^* \tag{5.19}$$

$$\frac{dT^*}{dt} = \left(1-\min 1\left\{, \max\left\{0, \frac{kVT(\lambda_1-\lambda_2)}{B_2}\right\}\right\}\right)\frac{kVT}{(1+\alpha_1 V)} - \beta T^* - \rho T^* \tag{5.20}$$

$$\frac{dV}{dt} = \left(1-\min\left\{1, \max\left\{0, \frac{N\beta T^*}{B_2}(\lambda_4-\lambda_3)\right\}\right\}\right) - \gamma V \tag{5.21}$$

$$\frac{dW}{dt} = \left(\min\left\{1, \max\left\{0, \frac{N\beta T^*}{B_2}(\lambda_4-\lambda_3)\right\}\right\}\right) N\beta T^* - \gamma W \tag{5.22}$$

Applying (5.18) in (5.11) and differentiate w.r.t to T, T^*, V, W we have;

$$\frac{\partial H}{dT} = -1-\lambda_1\left(\alpha + r\left(1-\frac{T}{T_{\max}}\right)-\left(1-\min\left\{1,\max\left\{0,\frac{kVT}{(1+\alpha_1 V)B_1}(\lambda_1-\lambda_2)\right\}\right\}\right)\right)\frac{kVT}{(1+\alpha_1 V)}+$$
$$\lambda_2\left(1-\min\left\{1,\max\left\{0,\frac{kVT}{(1+\alpha_1 V)B_1}(\lambda_1-\lambda_2)\right\}\right\}\right)\frac{kVT}{(1+\alpha_1 V)} \quad (5.23)$$

$$\frac{\partial H}{dT^*} = 1\left(-\lambda_1\frac{T}{T_{\max}}\right)+\rho\lambda_1-\lambda_2(\beta+\rho)+\lambda_3\left(1-\min\left\{1,\max\left\{0,\frac{N\beta T^*(\lambda_4-\lambda_3)}{B_2}\right\}\right\}\right)N\beta$$
$$+\lambda_4\left(1-\min\left\{1,\max\left\{0,\frac{N\beta T^*(\lambda_4-\lambda_3)}{B_2}\right\}\right\}\right)N\beta \quad (5.24)$$

$$\frac{\partial H}{dV} = -\lambda_3\left(1-\min\left\{1,\max\left\{0,\frac{N\beta T^*(\lambda_4-\lambda_3)}{B_2}\right\}\right\}\right)$$
$$+\lambda_4\left(1-\min\left\{1,\max\left\{0,\frac{N\beta T^*(\lambda_4-\lambda_3)}{B_2}\right\}\right\}\right)\frac{kT(1-V)}{(1+\alpha_1 V)^2}-\lambda_3\gamma \quad (5.25)$$

$$\frac{\partial H}{dW} = \lambda_4\gamma \quad (5.26)$$

With

$$T(0)=T_0, T^*(0)=T_0^8, V(0)=V_0, W(0)=W_0, \lambda_i(tf)=0 \qquad for \qquad i=1,2,3,4 \quad (5.27)$$

However, we solve the adjoin equations for problem (5.5-5.8)
Using Pontryagin's Maximum Principle;

$$\frac{d\lambda_1}{dt}=\frac{\partial H}{\partial T}, \quad \frac{d\lambda_2}{dt}=\frac{-\partial H}{\partial T^*} \quad \frac{d\lambda_3}{dt}=\frac{\partial H}{\partial V} \quad \frac{d\lambda_3}{dt}=\frac{-\partial H}{\partial W} \quad (5.28)$$

Applying equation (5.28), we have

$$\frac{d\lambda_1}{dt} = -\left(1-\lambda_1\left(\alpha+r\left(1-\frac{T}{T_{\max}}\right)\right)\right)-(1-u_1)\frac{kV}{1+\alpha_1 V}+\lambda(1-u_1)\frac{kV}{1+\alpha_1 V} \quad (5.29)$$

$$\frac{d\lambda_2}{dt} = -\left(1-\lambda_1\left(\frac{T}{T_{\max}}\right)+\rho\lambda_1-\lambda_2(\beta+\rho)+\lambda_3(1-u_2)N\beta\right)+\lambda_4 u_2 N\beta \quad (5.30)$$

$$\frac{d\lambda_3}{dt} = -(-\lambda_1(1-u_2)\frac{kT(1-V)}{(1+\alpha_1 V)^2}+\lambda_2(1-u_1)\frac{kT(1-V)}{(1+\alpha_1 V)^2}-\gamma w \quad (5.31)$$

$$\frac{d\lambda_4}{dt} = -(-\lambda_4 w)=\lambda_4 W \quad (5.32)$$

with transversalty conditions

$$\lambda_i(t_f)=0; \qquad i=1...4 \quad (5.33)$$

Theorem 5.1: given optimal controls u_1^*, u_2^* and solution $\dot{T}, \dot{T}^*, \dot{V}$ *and* $\dot{W}$ of the corresponding state system of problem (5.9), there exists adjoint variable $\lambda_1, \lambda_2, \lambda_3$ *and* λ_4 satisfying the Equations (5.33). However, the optimal control is given by (5.18)

Proof: the adjoint equations and transversality conditions can be obtained by using Pontryagin maximum principle such as;

$$\left.\begin{aligned} \lambda_1^1 &= -\frac{\partial H}{\partial T},\ \lambda_1(t_f) = 0; & \lambda_2^1 &= -\frac{\partial H}{\partial T^*},\ \lambda_2(t_f) = 0 \\ \lambda_3^1 &= -\frac{\partial H}{\partial v},\ \lambda_3(t_f) = 0; & \lambda_4^1 &= -\frac{\partial H}{\partial T},\ \lambda_4(t_f) = 0 \end{aligned}\right\} \tag{5.34}$$

The optimal control u_1^* *and* u_2^* can be solved from the optimality conditions

$$\frac{\partial H}{\partial u_1} = 0, \qquad \frac{\partial H}{\partial u_2} = 0 \tag{5.35}$$

That is

$$\left.\begin{aligned} \frac{\partial H}{\partial u_1} &= -B_1 u_1 + (\lambda_1 + \lambda_2)\frac{kVT}{1+\alpha_1 V} = 0 \\ \frac{\partial H}{\partial u_2} &= -B_2 u_2 + (\lambda_4 + \lambda_3) N\beta T^8 = 0 \end{aligned}\right\} \tag{5.36}$$

By the bounds in u of the controls, it is easy to obtain u_1^* *and* u_2^* in the form of Equation (5.10)

5.3Existence of an Optimal Control Pair

The existence of the optimal control pair can be obtained using a result by [47].

Theorem 5.2: Consider the control problem in (5.5-5.8) there exist an optimal control pair $(u_1^*, u_2^*) \in u$ such that $J(u_1^*, u_2^*) = \max_{(u_1,u_2)\in} J(u_1, u_2)$ (5.37)

To proof the theorem in (5.37) an existence result is used and the following properties are checked:

(1) The set of controls are corresponding to state variables is nonempty
(2) The control U set is convex and closed
(3) The right hand side of the state system is bounded by a linear function in the state and control variables.
(4) The integrand of the objective functional is concave on U.
(5) There exists constants $c_1, c_2 > 0$, and $B > 0$ such that the integrand $L(x, z, u_1, u_2)$ of the objective functional satisfies

$$L(x, z, u_1, u_2) \le c_2 - c_1(|u_1|^2 + |u_2|^2)^{B/2} \tag{5.38}$$

In order to verify these conditions, we use a result by [43]to give existence of solutions of system equation (5.5-5.8), which gives condition 1. The control set is convex and closed by definition, which gives condition 2. Since our state system is bilinear in u_1,u_2, the right hand sides of system equations (5.5-5.8), satisfies condition 3, using the boundedness of the solutions. The integrand of our objective functional is concave.
Now to the last condition needed.
Where c_2 depends on the upper bound on x, and $c_1 > 0$ since $B_1, B_2 > 0$. Hence, there exists an optimal control pair.

5.4 Uniqueness of the Optimality System

Theorem 5.3: The function $u^*(c) = \min(\max(c, a), b)$ is Lipschitz continuous in c, where a<b are some fixed positive constants.

Proof: consider c1, c2 real number and a,b are fixed positive constants. We will show that the Lipschitz continuity holds in all possible cases for max(c,a). Similar arguments hold for $\min(\max(c, a), b)$ as well.

(1) $c_1 \geq a,\ c_2 \geq a : \left|\max(c_1,a) - \max(c_2,a)\right| = \left|(c_1 - c_2)\right|$

(2) $c_1 \geq a,\ c_2 \leq a : \left|\max(c_1,a) - \max(c_2,a)\right| = \left|(c_1 - a)\right| \leq \left|(c_1 - c_2\right|$

(3) $c_1 \leq a,\ c_2 \geq a : \left|\max(c_1,a) - \max(c_2,a)\right| = (a - c_2) \leq \left|(c_1 - c_2)\right|$

(4) $c_1 \leq a,\ c_2 \leq a : \left|\max(c_1,a) - \max(c_2,a)\right| = \left|(a - a)\right| = 0 \leq \left|(c_1 - c_2)\right|$

Hence $\left|\max(c_1,a) - \max(c_2,a)\right| = \left|(c_1 - c_2)\right|$ and we have Lipschitz continuity of u* in c.

Theorem 5.4. For sufficiently small final time (t_f), bounded solutions to the optimality systems, (5.19-5.22) and (5.29-5.32) are unique.

Proof: With assumption in [49]; suppose $(T, T^*, V, W, \lambda_1, \lambda_2, \lambda_3, \lambda_5)$ and $(\overline{T}, \overline{T}^*, \overline{V}, \overline{W}, \overline{\lambda}_1, \overline{\lambda}_2, \overline{\lambda}_3, \overline{\lambda}_5)$ are two different solutions of our optimality system (5.19-5.22) and (5.29-5.32).

Let

$T = e^{\lambda t} p,\ , T^* = e^{\lambda t} p^*, V = e^{\lambda t} q, W = e^{\lambda t} x,\ \lambda_1 = e^{-\lambda t} w, \lambda_2 = e^{-\lambda t} z, \lambda_3 = e^{-\lambda t} v, \lambda_4 = e^{-\lambda t} y$ and

$T = e^{\lambda t} \overline{p},\ , T^* = e^{\lambda t} \overline{p}^*, \overline{V} = e^{\lambda t} \overline{q}, \overline{W} = e^{\lambda t} \overline{x},\ \overline{\lambda}_1 = e^{-\lambda t} \overline{w}, \overline{\lambda}_2 = e^{-\lambda t} \overline{z}, \overline{\lambda}_3 = e^{-\lambda t} \overline{v}, \overline{\lambda}_4 = e^{-\lambda t} \overline{y}$

Where $\lambda > 0$ is to be chosen. Further we let

$$u_1^* = \min\left\{\max\left\{a_1, \frac{(w-z)ke^{\lambda t}}{(1+\alpha_1 v)B_1} pq\right\}, b_1\right\} \tag{5.40}$$

$$u_2^* = \min\left\{\max\left\{a_2, \frac{(v-y)N\beta e^{\lambda t}}{B_2} p^*\right\}, b_2\right\} \tag{5.41}$$

and

$$\overline{u}_1^* = \min\left\{\max\left\{a_1, \frac{(\overline{w}-\overline{z})ke^{\lambda t}}{(1+\alpha_1 v)B_1} pq\right\}, b_1\right\} \tag{5.42}$$

$$\overline{u}_2^* = \min\left\{\max\left\{a_2, \frac{(\overline{v}-\overline{y})N\beta e^{\lambda t}}{B_2} p^*\right\}, b_2\right\} \tag{5.43}$$

Now we substitute $T = e^{\lambda t} p$ into the first ODE of (5.28) and get

$$p' + \lambda p = e^{-\lambda t} s - \alpha p + rp\left(1 - \frac{p + p^*}{T_{\max}}\right) - \left(1 - \min\left\{\max\left\{a_1, \frac{(\overline{w}-\overline{z})ke^{\lambda t}}{(1+\alpha_1 v)B_1} pq\right\}, b_1\right\}\right)\frac{ke^{\lambda t} pq}{1+\alpha_1 vq} + p^* \tag{5.44}$$

Similarly,

$$\left.\begin{aligned} &T^* = e^{\lambda t} p^*; v = e^{\lambda t} q,\ W = e^{\lambda t} x,\ \ \lambda_1 = e^{\lambda t} w, \\ &\lambda_2 = e^{\lambda t} z; \lambda_3 = e^{\lambda t} v, \lambda_4 = e^{\lambda t} y \end{aligned}\right\} \tag{5.45}$$

We obtained

$$p^{*1} + \lambda p^* = \left(1 - \min\left\{\max\left\{a_1, \frac{(\overline{w}-\overline{z})ke^{\lambda t}}{(1+\alpha_1 v)B_1} pq\right\}, b_1\right\}\right)\frac{ke^{\lambda t} pq}{1+\alpha_1 vq} - (\beta + \rho)p^* \tag{5.46}$$

$$q' + \lambda q = \left(1 - \min\left\{\max\left\{a_1, \frac{(\overline{v}-\overline{y})N\beta e^{\lambda t}}{(1+\alpha_1 v)B_2} p^*\right\}, b_1\right\}\right)N\beta p^* - \gamma q \tag{5.47}$$

$$x' + \lambda x = \min\left\{\max\left\{a_1, \frac{(\overline{v}-\overline{y})N\beta e^{\lambda t}}{(1+\alpha_1 v)B_2} p^*\right\}, b_1\right\}N\beta p^* - \gamma q \tag{5.48}$$

$$-v'+\lambda v=-w\left(1-\min\left\{\max\left\{a_1,\frac{(\bar{v}-\bar{y})N\beta e^{\lambda t}}{2B_1}p^*\right\},b_1\right\}\right)ke^{\lambda t}+$$

$$z\left(1-\min\left\{\max\left\{a_1,\frac{(\bar{v}-\bar{y})N\beta e^{\lambda t}}{2B_1}p^*\right\},b_1\right\}\right)ke^{\lambda t}p-v\gamma \qquad (5.49)$$

$$-y'+\lambda y=-yw \qquad (5.50)$$

From equation (5.29-5.33), we obtain;

$$w'+\lambda w=e^{\lambda t}-w\left(\alpha+r\left(1-\frac{2p}{T_{\max}}\right)\right)-\left(1-\min\left\{\max\left\{a_1,\frac{(\bar{w}-\bar{z})ke^{\lambda t}pq}{(1+\alpha_1 v)B_1}\right\},b_1\right\}\right)\frac{kpq}{1+\alpha_1 Vq}$$

$$+z\left(1-\min\left\{\max\left\{a_1,\frac{(\bar{w}-\bar{z})ke^{\lambda t}pq}{(1+\alpha_1 V)B_1}\right\},b_1\right\}\right)\frac{kpq}{1+\alpha_1 vq} \qquad (5.51)$$

$$-z'+\lambda z=-w\left(\frac{e^{\lambda t}p}{T_{\max}}\right)+\rho w-z(\beta+\rho)+\left(1-\min\left\{\max\left\{a_1,\frac{\bar{v}-\bar{y}}{B_2}N\beta e^{\lambda t}p^*\right\},b_2\right\}\right)vN\beta+$$

$$yN\beta\left(1-\min\left\{\max\left\{a_1,\frac{\bar{v}-\bar{y}}{B_2}N\beta e^{\lambda t}p^*\right\},b_2\right\}\right) \qquad (5.52)$$

Now we subtract the equations for T and $\bar{T}$; T^* and $\bar{T}^*$; V and $\bar{V}$; W and $\bar{W}$;

$$\lambda_1 \text{ and } \bar{\lambda}_1;\lambda_2 \text{ and } \bar{\lambda}_2;\lambda_3 \text{ and } \bar{\lambda}_3;\lambda_4 \text{ and } \bar{\lambda}_4 \qquad (5.53)$$

Then multiply each equation by appropriate different of functions and integrate from 0 to t_f. Next we add all eight integral equations and will use estimates to obtain uniqueness. Using Theorem (5.3); we have

$$\left|u_1^*-\bar{u}_1^*\right|\le\frac{1}{(1+\alpha_1 V)B_1}\left|(w-z)\frac{ke^{\lambda t}pq}{1+\alpha_1 q}-(\bar{w}-\bar{z})\frac{ke^{\lambda t}p\bar{q}}{1+\alpha_1\bar{q}}\right| \qquad (5.54)$$

And $\left|u_2^*-\bar{u}_2^*\right|\le\frac{1}{B_2}\left|(v-y)N\beta e^{\lambda t}p^*-(\bar{v}-\bar{y})N\beta ke^{\lambda t}\bar{p}\right|$ (5.55)

One case of the estimate was illustrated using on equation $\left|u_1^*-\bar{u}_1^*\right|$ Hence, we obtain:

$$\frac{1}{2}(p-\bar{p})^2(t_f)+\lambda\int_0^{t_f}(p-\bar{p})^2dt\le\alpha\int_0^{t_f}|p-\bar{p}||p-\bar{p}|dt+$$

$$r\int_0^{t_f}\left|e^{\lambda t}p\left(1-\frac{p+p^*}{T_{\max}}\right)-e^{\lambda t}\bar{p}\left(1-\frac{\bar{p}+\bar{p}^*}{T_{\max}}\right)\right||p-\bar{p}|dt-$$

$$\int_0^{t_f}\left|u_1^*-\bar{u}_1^*\right||p-\bar{p}|dt+\rho\int_0^{t_f}\left|p^*-\bar{p}^*\right||p-p|dt\le c_1\int_0^{t_f}\Big[|p-\bar{p}|^2+\left|p^*-\bar{p}^*\right|^2+|q-\bar{q}|^2+|x-\bar{x}|^2+$$

$$|w-\bar{w}|^2\Big]dt+$$

$$c_2\int_0^{t_f}\Big[|p-\bar{p}|^2+\left|p^*-\bar{p}^*\right|^2+|q-\bar{q}|^2+|x-\bar{x}|^2dt \qquad (5.56)$$

Where c_1 and c_2 depend on the coefficients and the bounds on states and adjoints. Combining four of these estimates gives

$$\frac{1}{2}(p-\bar{p})^2(t_f)+\frac{1}{2}(p^*-\bar{p}^*)^2(t_f)+\frac{1}{2}(q-\bar{q})^2(0)+(w-\bar{w})^2\Big]dt+\frac{1}{2}(z-z)^2(0)+$$

$$\lambda\int_0^{t_f}\Big[|p-\bar{p}|^2+|p^*-\bar{p}^*|^2+|q-\bar{q}|^2+|w-\bar{w}|^2+|z-\bar{z}|^2\Big]dt\le\left(\tilde{C}_1+\tilde{C}_2e^{3\lambda t_f}\right)$$

$$\int_0^{t_f}\Big[|p-\bar{p}|^2+|p^*-\bar{p}^*|^2+|q-\bar{q}|^2+|w-\bar{w}|^2+|z-\bar{z}|^2\Big] \tag{5.57}$$

Thus from above equation we conclude that;

$$\left(\tilde{C}_1+\tilde{C}_2e^{3\lambda t_f}\right)\int_0^{t_f}\Big[|p-\bar{p}|^2+|p^*-\bar{p}^*|^2+|q-\bar{q}|^2+|w-\bar{w}|^2+|z-\bar{z}|^2\Big]dt\le 0 \tag{5.58}$$

Where $\tilde{C}_1,\tilde{C}_2$ depend on the coefficients and the bounds on P, P*,w,z

If we choose λ such that $\lambda>\tilde{C}_1+\tilde{C}_2$ and $t_f<\frac{1}{3\lambda}In\left(\frac{\lambda-\bar{C}_1}{\bar{C}_2}\right)$, then $p=\bar{p}$

$p^*=\bar{p}^*,\bar{w}=\bar{w};z=\bar{z}$. Hence the solution is unique for small time

6.1 Adomian Decomposition Technique (ADM)

An explicit construction of approximate non-perturbative solutions of the system (5.5-5.8) was examining using Adomian decomposition method .The equivalent canonical form of this system is as follows:

$$T(t)=T(0)+\int_0^x sdt+(r-\alpha)\int_0^x Tdt-\frac{r}{T_{\max}}\int_0^x T^2+\frac{r}{T_{\max}}\int_0^x TT^*dt-(1-u_1)k(1+\alpha_1)^{-1}\int_0^x Vdt.\int_0^x VTdt+$$

$$\rho\int_0^x T^*dt \tag{6.1}$$

$$T^*(t)=T^*(0)+(1-u_1)k(1+\alpha_1)^{-1}\int_0^x Vdt.\int_0^x VTdt-(\beta+\rho)\int_0^x T^*dt \tag{6.2}$$

$$V(t)=V(0)+(1-u_2)N\beta\int_0^x T^*dt-\gamma\int_0^x Vdt \tag{6.3}$$

$$W(t)=W(0)+u_2N\beta\int_0^x T^*dt-\gamma\int_0^x Wdt \tag{6.4}$$

Adomian decomposition method was used to obtain the solutions of equations (3.290-3.293) are considered being as the sum of the following series;

$$T=\sum_{n=0}^{\infty}T_n \tag{6.5}$$

$$T^*\sum_{n=0}^{\infty}T_n^* \tag{6.6}$$

$$V=\sum_{n=0}^{\infty}V_n \tag{6.7}$$

$$W=\sum_{n=0}^{\infty}W_n \tag{6.8}$$

Then we approximate the non-linear terms in the system as follows;

$$TT = \sum_{n=0}^{\infty}\left(A_n(T_0....T_n, \quad T_0....T_n)\right) \tag{6.9}$$

$$VT = \sum_{n=0}^{\infty}\left(B_n(V_0....V_n, \quad T_0....T_n)\right) \tag{6.10}$$

$$TT^* = \sum_{n=0}^{\infty}\left(C_n(T_0....T_n, \quad T_0^*....T_0^*)\right) \tag{6.11}$$

Where

$$A_n = \frac{1}{n!}\left[\frac{d^n\left(\sum_{m=0}^{\infty}T_m\lambda^m\right)\left(\sum_{m=0}^{\infty}T_m\lambda^m\right)}{d\lambda^m}\right]_{m=0} \tag{6.12}$$

$$B_n = \frac{1}{n!}\left[\frac{d^n\left(\sum_{m=0}^{\infty}V_m\lambda^m\right)\left(\sum_{m=0}^{\infty}T_m\lambda^m\right)}{d\lambda^m}\right]_{m=0} \tag{6.13}$$

$$C_n = \frac{1}{n!}\left[\frac{d^n\left(\sum_{m=0}^{\infty}T_m\lambda^m\right)\left(\sum_{m=0}^{\infty}T_m^*\lambda^m\right)}{d\lambda^m}\right]_{m=0} \tag{6.14}$$

The non-linear function A_n, B_n, C_n, are called Adomian's polynomials. Substituting equation (6.5-5.14) into (6.1-6.4) then we have;

$$\sum_{n=0}^{\infty}T_n = T(0) + sx + (r-\alpha)\int_0^x\sum_{n=0}^{\infty}T_n dt - \frac{r}{T_{\max}}\int_0^x\sum_{n=0}^{\infty}A_n dt + \frac{r}{T_{\max}}\int_0^x\sum_{n=0}^{\infty}C_n dt -$$

$$k(1-u_1)(1+\alpha_1)^{-1}\int_0^x\sum_{n=0}^{\infty}V_n dt\int_0^x\sum_{n=0}^{\infty}B_n dt + \rho\int_0^x\sum_{n=0}^{\infty}T_n^* dt \tag{6.15}$$

$$\sum_{n=0}^{\infty}T_n^* = T^*(0) + k(1-u_1)(1+\alpha_1)^{-1}\int_0^x\sum_{n=0}^{\infty}V_n dt\int_0^x\sum_{n=0}^{\infty}B_n dt - (\beta+\rho)\int_0^x\sum_{n=0}^{\infty}T_n^* dt \tag{6.16}$$

$$\sum_{n=0}^{\infty}V_n = V(0) + (1-u_2)N\beta\int_0^x\sum_{n=0}^{\infty}T_n^* dt - \gamma\int_0^x\sum_{n=0}^{\infty}V_n dt \tag{6.17}$$

$$\sum_{n=0}^{\infty}W_n = W(0) + u_2 N\beta\int_0^x\sum_{n=0}^{\infty}T_n^* dt - \gamma\int_0^x\sum_{n=0}^{\infty}W_n dt \tag{6.18}$$

From equation (6.15-6.18) we define the following scheme:

$$T_0 = T(0) + st \tag{6.19}$$

$$T_0^* = T^*(0) \tag{6.20}$$

$$V_0 = V(0) \tag{6.21}$$

$$W_0 = W(0) \tag{6.22}$$

$$\sum_{n=0}^{\infty}T_n=(r-\alpha)\int_0^x\sum_{n=0}^{\infty}T_n dt-\frac{r}{T_{\max}}\int_0^x\sum_{n=0}^{\infty}A_n dt+\frac{r}{T_{\max}}\int_0^x\sum_{n=0}^{\infty}C_n dt-$$

$$k(1-u_1)(1+\alpha_1)^{-1}\int_0^x\sum_{n=0}^{\infty}V_n dt\int_0^x\sum_{n=0}^{\infty}B_n dt+\rho\int_0^x\sum_{n=0}^{\infty}T_n^* dt \qquad (6.23)$$

$$(for\ \ n\geq 0)$$

$$\sum_{n=0}^{\infty}T_n^*=k(1-u_1)(1+\alpha_1)^{-1}\int_0^x\sum_{n=0}^{\infty}V_n dt\int_0^x\sum_{n=0}^{\infty}B_n dt-(\beta+\rho)\int_0^x\sum_{n=0}^{\infty}T_n^* dt \qquad (6.24)$$

$$(for\ \ n\geq 0)$$

$$\sum_{n=0}^{\infty}V_n=(1-u_2)N\beta\int_0^x\sum_{n=0}^{\infty}T_n^* dt-\gamma\int_0^x\sum_{n=0}^{\infty}V_n dt \qquad (6.25)$$

$$(for\ \ n\geq 0)$$

$$\sum_{n=0}^{\infty}W_n=u_2N\beta\int_0^x\sum_{n=0}^{\infty}T_n^* dt-\gamma\int_0^x\sum_{n=0}^{\infty}W_n dt \qquad (6.26)$$

$$(for\ \ n\geq 0)$$

Using Equation (6.9-6.14). Some of the Adomian polynomials can be obtained as

follows;

$$F(t)=T^2 \qquad (6.27)$$

We first set

$$T=\sum_{n=0}^{\infty}T_n \qquad (6.28)$$

Substituting (6.28) into (6.27)gives

$$F(t)=(T_0+T_1+T_2+T_3+T_4+T_5+....)^2 \qquad (6.29)$$

Expanding the expression at the right-hand side gives

$$F(t)=T_0^2+2T_0T_1+2T_0T_2+T_1^2+2T_0T_3+2T_1T_2+... \qquad (6.30)$$

The expansion in equation (6.30) can be rearranged by grouping all terms with the sum of the

subscripts of the components of the same. This means that we can rewrite equation (6.30) as

$$F(t)=T_0^2+2T_0T_1+2T_0T_2+T_1^2+2T_0T_3+2T_0T_2+2T_0T_4+2T_1T_2+T_2^2+$$
$$2T_0T_5+2T_1T_4+2T_2T_3+.... \qquad (6.31)$$

This gives Adomian polynomials for Equation (3.316) by

$$\left.\begin{aligned}A_0&=T_0^2\\A_1&=2T_0T_1\\A_2&=2T_0T_2+T_1^2\\A_3&=2T_0T_3+2T_0T_2\\A_4&=2T_0T_4+2T_1T_2+T_2^2\\A_5&=2T_0T_5+2T_1T_4+2T_2T_3\end{aligned}\right\} \qquad (6.32)$$

$$F(t)=VT \qquad (6.33)$$

We first set

$$V = \sum_{n=0}^{\infty} V_n \tag{6.34}$$

$$T = \sum_{n=0}^{\infty} T_n \tag{6.35}$$

Substituting (6.34 and 6.35) into Equation (6.33) yields

$$F(T) = (V_0 + V_1 + V_2 + V_3 + V_4 + V_5 + ...) \times (T_0 + T_1 + T_2 + T_3 + T_4 + T_5 + ...) \tag{6.36}$$

Multiplying the two factors gives

$$F(t) = V_0T_0 + T_0V_1 + V_0T_1 + T_0V_2 + V_0T_2 + T_1V_1 + T_2V_0 + T_0V_3 + T_1V_2 + T_2V_2 + T_3V_0 + T_0V_4 + V_0T_4 + T_1V_3 + V_1T_3 + V_2T_2 + ... \tag{6.37}$$

Collecting all terms with the same sum of subscripts of the component T_n, we can rewritten Equation (6.37) in the form

$$F(t) = T_0V_1 + V_0T_1 + T_0V_2 + T_1V_1 + T_2V_0 + T_0V_3 + T_1V_2 + T_2V_1 + T_3V_0 + T_0V_4 + T_1V_3 + T_2V_2 + T_3V_1 + T_4V_0 + ... \tag{6.38}$$

Consequently, the Adomian polynomials are given by

$$\left.\begin{aligned} B_0 &= T_0V_0 \\ B_1 &= T_0V_1 + V_0T_1 \\ B_2 &= T_0V_2 + T_1V_1 + T_2V_0 \\ B_3 &= T_0V_3 + T_1V_2 + T_2V_1 + T_3V_0 \\ B_4 &= T_0V_4 + T_1V_3 + T_2V_2 + T_3V_1 + T_4V_0 \end{aligned}\right\} \tag{6.39}$$

$$T(t) = TT^* \tag{6.40}$$

We first set

$$T = \sum_{n=0}^{\infty} T_n \tag{6.41}$$

$$T^* = \sum_{n=0}^{\infty} T_n^* \tag{6.42}$$

Substituting (6.42 and 6.41) into Equation (6.40) yields

$$F(T) = (T_0 + T_1 + T_2 + T_3 + T_4 + T_5 + ...) \times (T^*_0 + T^*_1 + T^*_2 + T^*_3 + T^*_4 + T_5 + ...) \tag{6.43}$$

Multiplying the two factors gives

$$F(T) = T_0T_0^* + T_0^*T_1 + T_0T_1^* + T_0^*T_2 + T_1^*T_1 + T_2^*T_0 + T_0^*T_3 + T_1^*T_2 + T_2^*T_1 + T_3^*T_0 + T_0^*T_4 + T_0T_4^* + T_1^*T_3 + T_1T_3^* + T_2T_2^* + ... \tag{6.44}$$

Colleting all terms with the same sum of subscripts of the components T_n, we can rewritten equation (6.44) in the form

$$F(T) = T_0T_0^* + T_0^*T_1 + T_0T_1^* + T_0^*T_2 + T_1^*T_1 + T_2^*T_0 + T_0^*T_3 + T_1^*T_2 + T_2^*T_1 + T_3^*T_0 + T_0^*T_4 + T_1^*T_3 + T_2^*T_2 + T_3^*T_1 + T_4^*T_0 + ... \tag{6.45}$$

Consequently, the Adomian polynomials, are given by

$$\left.\begin{aligned} C_0 &= T_0T_0^* \\ C_1 &= T_0^*T_1 + T_0T_1^* \\ C_2 &= T_0^*T_2 + T_1^*T_1 + T_2^*T_0 \\ C_3 &= T_0^*T_3 + T_1^*T_2 + T_2^*T_1 + T_3^*T_0 \\ C_4 &= T_0^*T_4 + T_1^*T_3 + T_2^*T_2 + T_3^*T_1 + T_4^*T_0 \end{aligned}\right\} \tag{6.46}$$

IV. NUMERICAL RESULTS AND DISCUSSION

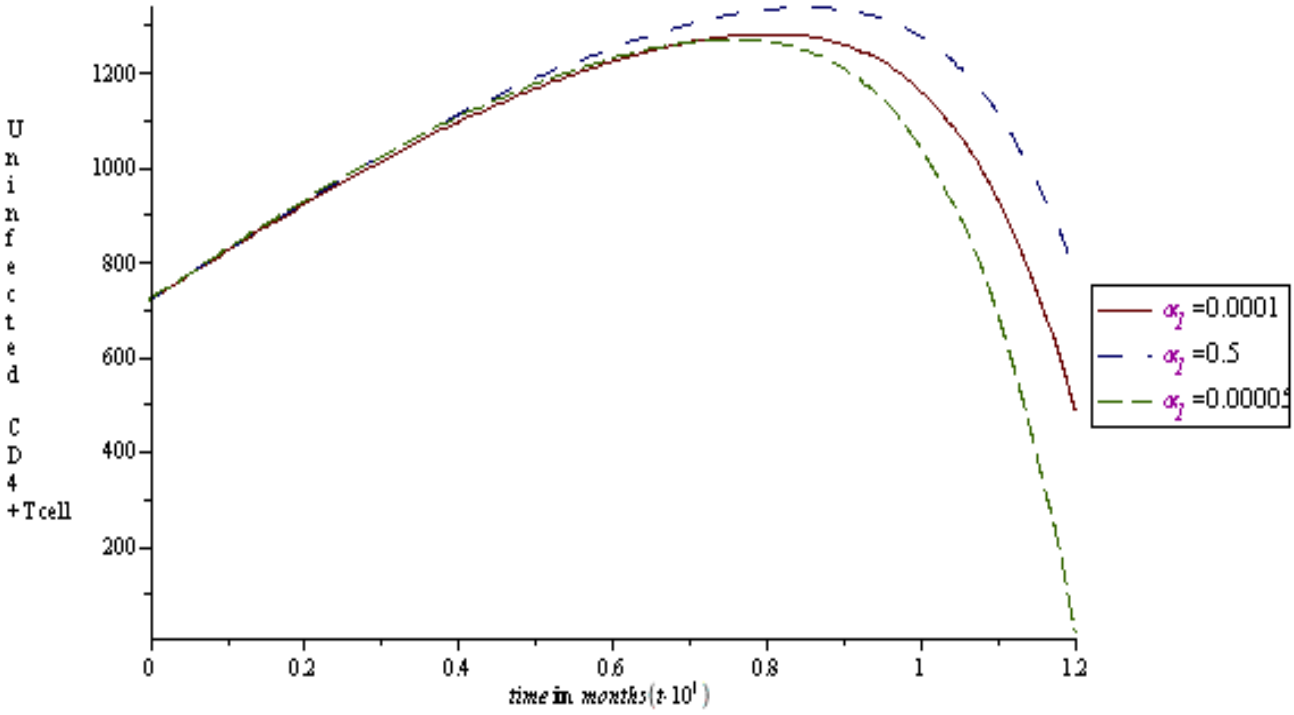

Fig 1: Uninfected CD4$^+$T-cell count with various α_1 and without treatment against time

when $s = 15; \beta = 0.000024; \mu_1 = 0.02; \sigma = 500; c = 2.4; \mu_3 = 0.26; q = 0.1;$
$\pi = 0.01; \rho = 0.05; \delta = 0.005; \gamma = 0.01; \mu_2 = 0.40; \alpha = 0.89$

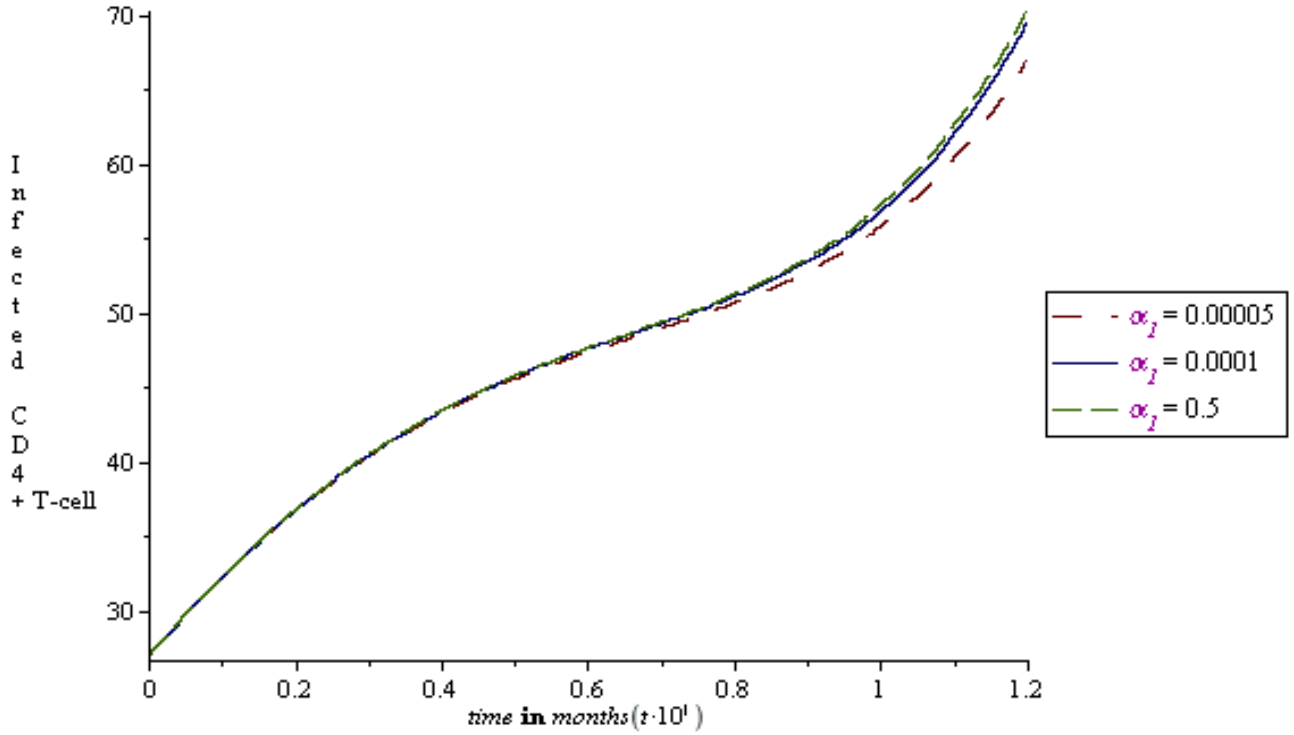

Fig 2: Infected CD4$^+$T-cell count with various α_1 and without treatment against time

when $s = 15; \beta = 0.000024; \mu_1 = 0.02; \sigma = 500; c = 2.4; \mu_3 = 0.26; q = 0.1;$
$\pi = 0.01; \rho = 0.05; \delta = 0.005; \gamma = 0.01; \mu_2 = 0.40; \alpha = 0.89$

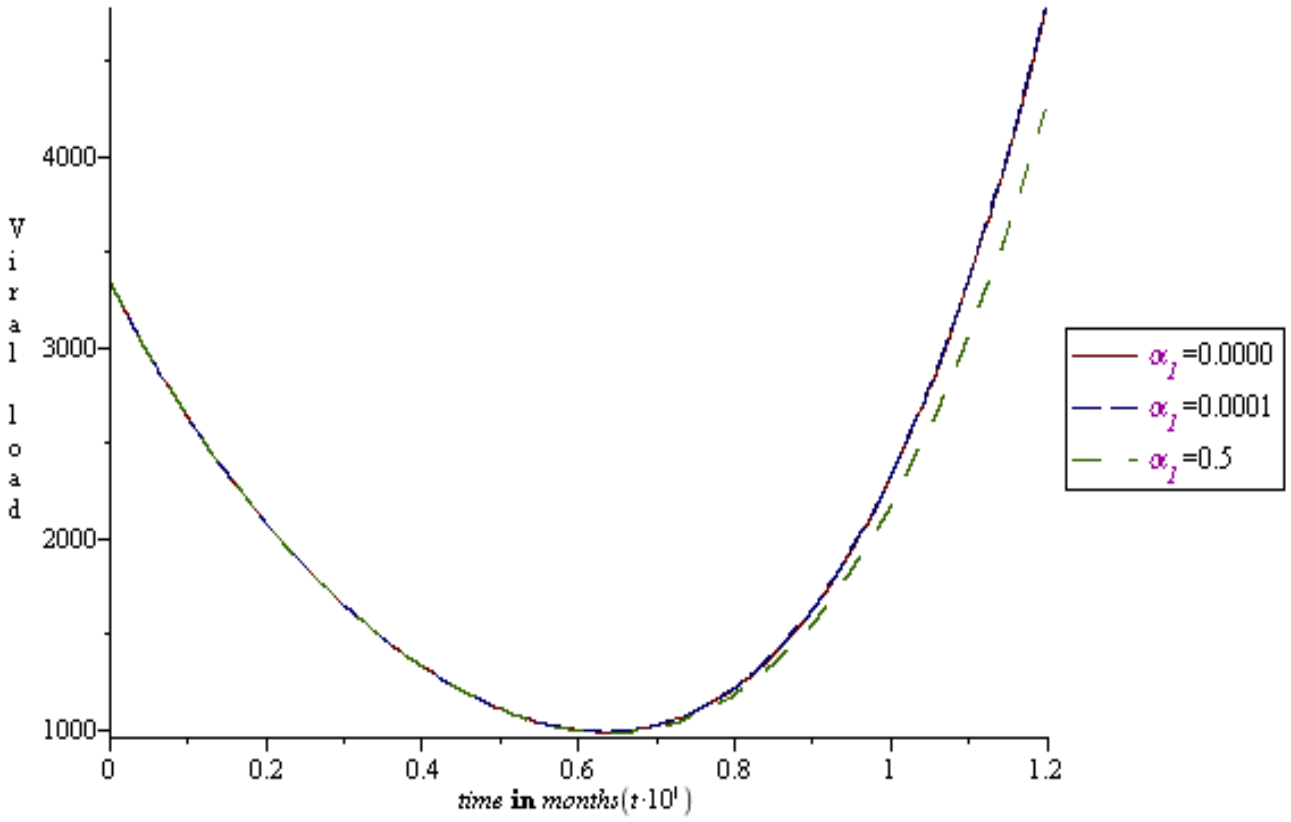

Fig 3: Viral particles with various α_1 and without treatment against time

when $s = 15; \beta = 0.000024; \mu_1 = 0.02; \sigma = 500; c = 2.4; \mu_3 = 0.26; q = 0.1;$
$\pi = 0.01; \rho = 0.05; \delta = 0.005; \gamma = 0.01; \mu_2 = 0.40; \alpha = 0.89$

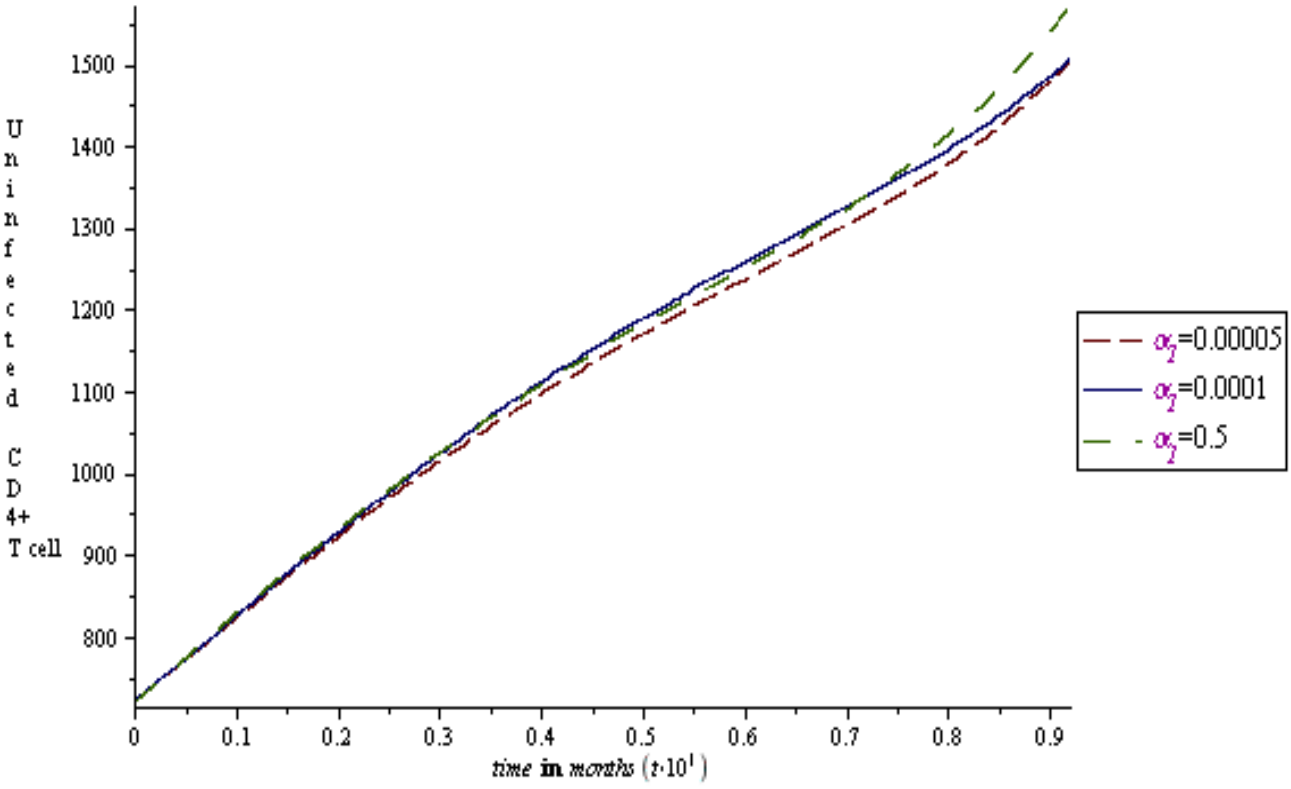

Fig 4: Uninfected CD4$^+$T-cell count with various α_1 and treatment against time

when $s = 15; \beta = 0.000024; \mu_1 = 0.02; \sigma = 500; c = 2.4; \mu_3 = 0.26; q = 0.1;$
$\pi = 0.01; \rho = 0.05; \delta = 0.005; \gamma = 0.01; \mu_2 = 0.40; \alpha = 0.89$

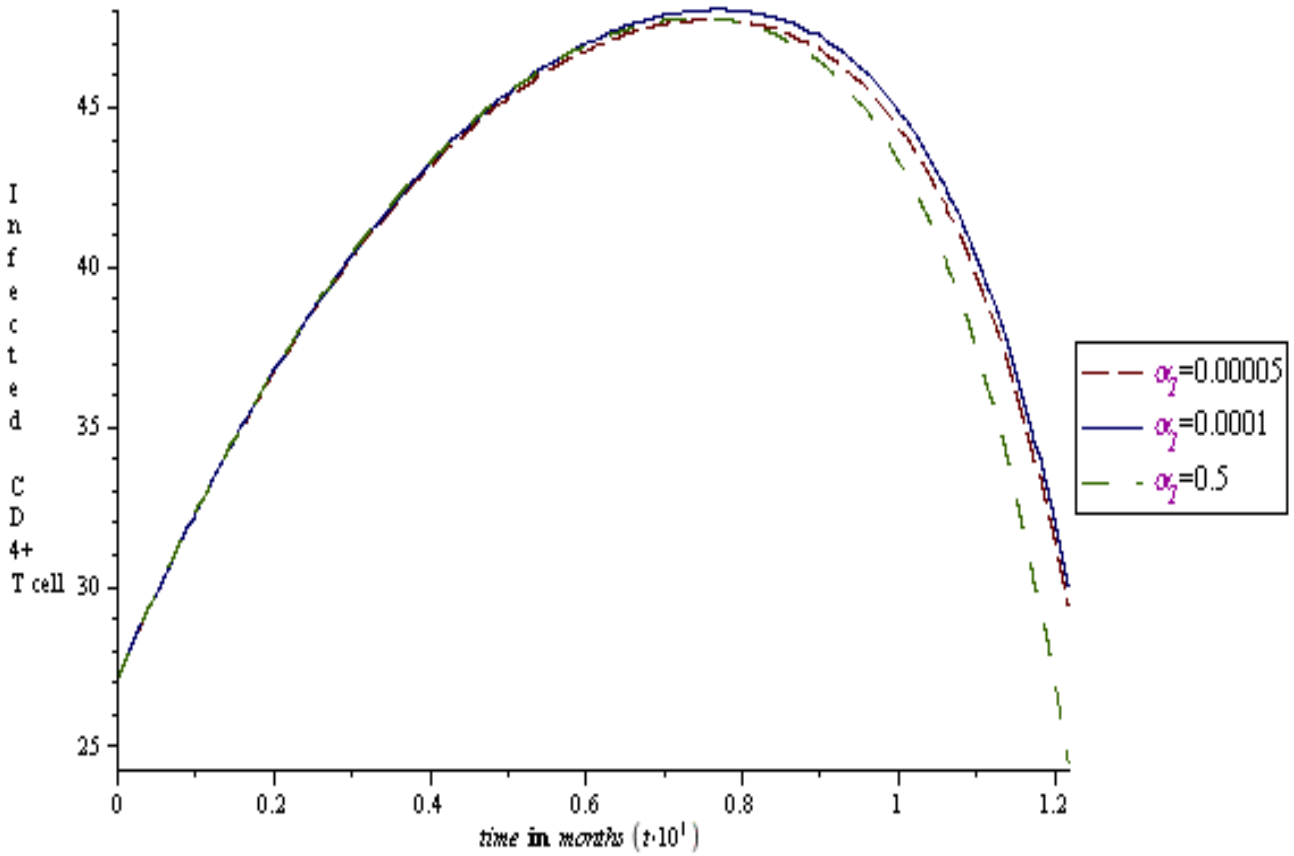

Fig 5: Infected CD4$^+$T-cell count with various α_1 and treatment against time

when $s = 15; \beta = 0.000024; \mu_1 = 0.02; \sigma = 500; c = 2.4; \mu_3 = 0.26; q = 0.1;$
$\pi = 0.01; \rho = 0.05; \delta = 0.005; \gamma = 0.01; \mu_2 = 0.40; \alpha = 0.89$

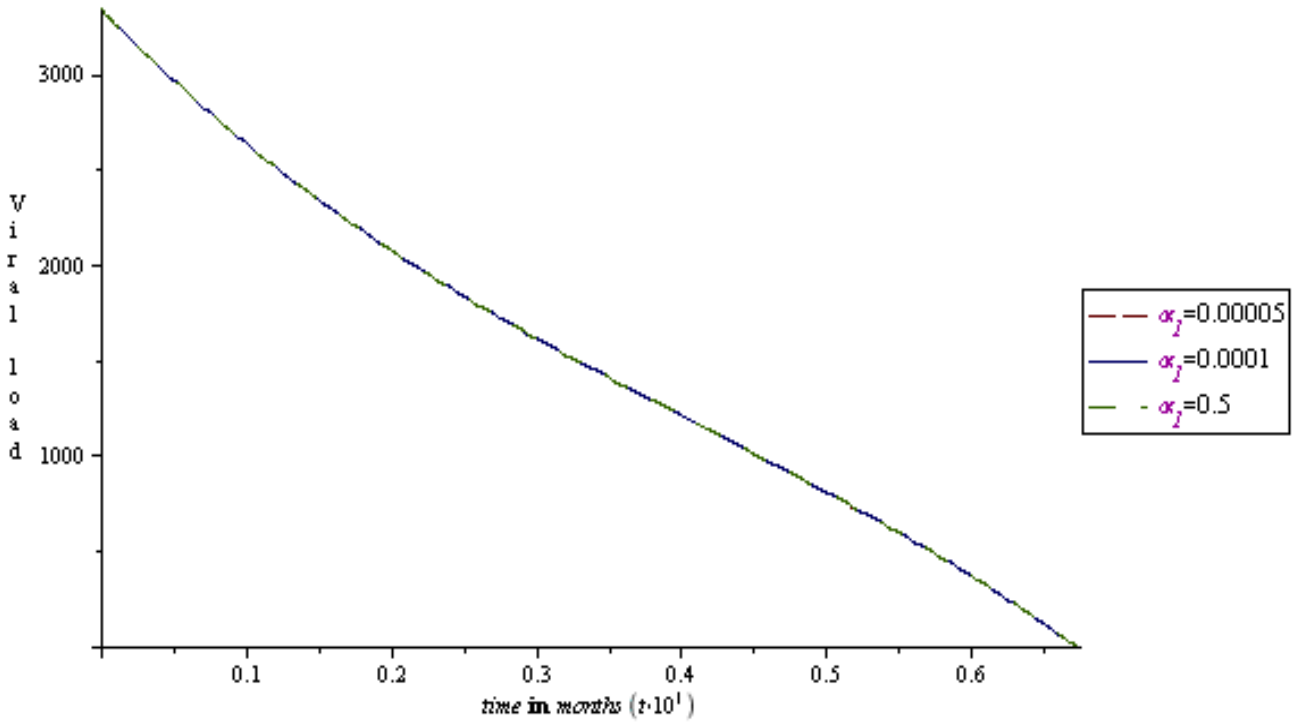

Fig 6: Infectious viral particle with various α_1 and treatment against time

when $s = 15; \beta = 0.000024; \mu_1 = 0.02; \sigma = 500; c = 2.4; \mu_3 = 0.26; q = 0.1;$
$\pi = 0.01; \rho = 0.05; \delta = 0.005; \gamma = 0.01; \mu_2 = 0.40; \alpha = 0.89$

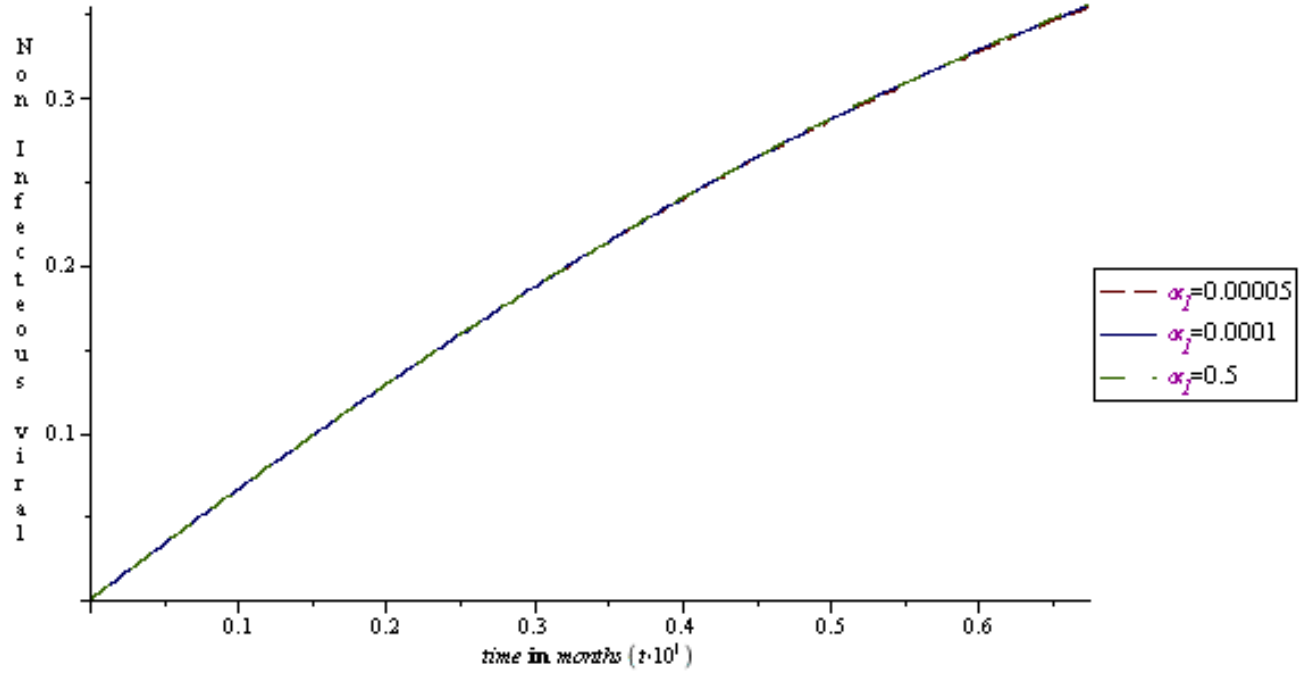

Fig 7: Non- infectious viral particle with various α_1 and treatment against time

when $s = 15; \beta = 0.000024; \mu_1 = 0.02; \sigma = 500; c = 2.4; \mu_3 = 0.26; q = 0.1;$
$\pi = 0.01; \rho = 0.05; \delta = 0.005; \gamma = 0.01; \mu_2 = 0.40; \alpha = 0.89$

V. CONCLUTIONS

In this paper, the study of the behavior of the mathematical model adopted to describe the HIV infection provides an explanation for the viral sudden deviation during the suppression of HIV. Adomian Decomposition Method (ADM) was also employed to compute an approximation to the solution of the non-linear system of differential equations governing the problem. Maple is used to carry out the computations and graphical results were presented to illustrate the solution.

Appendix

```
### ADOMIAN DECOMPOSITION METHOD ###
>restart:a:=array(0..50):x:=array(0..50):y:=array(0..50):APROX:=array(0..50):y[0]:=
0.0:x[0]:=0.0:h:=0.1:M:=8:N:=10:a[0]:=719.57:m:=0:beta:=0.3:alpha:=0.02:r:=3.0:g:=2
.4:k:=0.000027:T[max]:=1500.0:rho:=0.01:s:=15:b[0]:=27:c[0]:=3341:d[0]:=0:u[1]:=0.0
083:u[2]:=0.0083:U0:=s*t:U1:=0.0:U2:=0.0:U3:=0.0:T0:=(a[m]+t*diff(a[m],t))+(U0):I0:
=(b[m]+t*diff(b[m],t))+(U1):v0:=(c[m]+t*diff(c[m],t))+(U2):w0:=(d[m]+t*diff(d[m],t)
)+(U3):A0:=T0^2:B0:=v0*T0:C0:=T0*I0:T1:=alpha*int(T0,t=x[m]..t)+(r*int(T0,t=x[m]..t
))((r/T[max])*int(A0,t=x[m]..t))((r/T[max])*int(C0,t=x[m]..t))((1u[1])*k*int(B0,t=x
[m]..t))+(rho*int(I0,t=x[m]..t)):I1:=((1- u[1])*
k*int(B0,t=[x]..t))((beta+rho)*int(I0,t=x[m]..t)):v1:=(1u[2])*N*beta*int(I0,t=x[m].
.t)g*int(v0,t=x[m]..t):w1:=u[2]*N*beta*int(I0,t=x[m]..t)g*int(w0,t=x[m]..t):A1:=2*T
0*T1:B1:=v0*T1+v1*T0:C1:=T0*I1+T1*I0:T2:=alpha*int(T1,t=x[m]..t)+(r*int(T1,t=x[m]..
t))-((r/T[max])*int(A1,t=x[m]..t))-((r/T[max])*int(C1,t=0..t))-((1-u[1])*k*int(B1,t
=x[m]..t))+(rho*int(I1,t=x[m]..t)):I2:=((1-u[1])*k*int(B1,t=x[m]..t))-((beta+rho)*i
nt(I1,t=0..t)):v2:=(1-u[2])*N*beta*int(I1,t=x[m]..t)g*int(v1,t=x[m]..t):w2:=u[2]*N*
beta*int(I1,t=x[m]..t)g*int(w1,t=x[m]..t):A2:=2*T0*T2+T1^2:B2:=v0*T2+v1*T1+v2*T0:C2
:=T0*I2+T1*I1+T2*I0:T3:=alpha*int(T2,t=x[m]..t)+(r*int(T2,t=x[m]..t))((r/T[max])*in
t(A2,t=x[m]..t))((r/T[max])*int(C2,t=0..t))((1u[1])*k*int(B2,t=x[m]..t))+(rho*int(I
2,t=x[m]..t)):I3:=((1u[1])*k*int(B2,t=x[m]..t))((beta+rho)*int(I2,t=0..t)):v3:=(1-u
[2])*N*beta*int(I2,t=x[m]..t)g*
int(w2,t=x[m]..t):w3:=u[2]*N*beta*int(I2,t=x[m]..t)-g*int(v2,t=x[m]..t):A3:=2*T0*T3
+2*T1*T2:B3:=v0*T3+v1*T2+v2*T1+v3*T0:C3:=T0*I3+T1*I2+T2*I1+T3*I0:T4:=-alpha*int(T3,
t=x[m]..t)+
(r*int(T3,t=x[m]..t))-((r/T[max])*int(A3,t=x[m]..t))-((r/T[max])*int(C3,t=0..t))-((
1-u[1])*k*int(B3,t=x[m]..t))+
(rho*int(I3,t=x[m]..t)):I4:=((1-u[1])*k*int(B3,t=x[m]..t))-((beta+rho)*int(I3,t=0..
t)):v4:=(1- u[2])*N*beta*
```

```
int(I3,t=x[m]..t)- g*int(v3,t=x[m]..t):w4:=u[2]*
N*beta*int(I3,t=x[m]..t)-g*int(w3,t=x[m]..t):
Eq:=T0+T1+T2+T3+T4:Eq2:=I0+I1+I2+I3+I4:Eq3:=v0+v1+v2+v3+v4:Eq4:=w0+w1+w2+w3+w4:
```

$$T = 719.57 + 1030.113124t + 34.97820628t^2 - 719.6610060t^3 - 41.31829380t^4 + ...$$

$$T^* = 27 + 56.00465683t - 39.08106622t^2 - 6.185455807t^3 + 11.57469526t^4 + ...$$

$$V = 3341 - 7937.173347t + 9608.980705t^2 - 7735.947760t^3 + 4631.970360t^4 + ...$$

$$w = .6733470426t - .1097224390t^2 - .2370917650t^3 + .1037093337t^4 + ...$$

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