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Supercritical Solutions of the Stationary Positive Forced KdV Equation

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***Abstract*— In this paper we consider stationary Forced KdV equation with positive Forcing term. The supercritical solitary wave solutions of the stationary Forced KdV equation are obtained. In order to obtain the solutions the domain of the problem has been divided in to three parts; the left, the middle and the right parts. The solution on the left and the right parts are obtained by an analytical method. The solution on the middle part is expressed in the terms of Weierstarss elliptic function. We have designed computer programs using Mathematica to produce the solutions. The complete solution was found by matching the solutions of all the three parts. We have found out that there are four different solutions according to the values of the phase shift. Only one solution is positive. Further research can be carried out for negative forcing terms.**

***Index Terms*— Stationary Forced KdV Equation; Supercritical solution;. positive Forcin**

I. INTRODUCTION

The forced KdV equation is a first –order approximation of a long nonlinear surface wave in a channel flow of an in viscid fluid of constant density over a bump or a dent. This equation is normally written in the form (Gong and Shen, 1994)

$$\eta_t + \lambda\eta_x + 2\alpha\eta\eta_x + \beta\eta_{xxx} = f_x(x) \qquad -\infty < x < \infty \quad t > 0 \quad (1)$$

where $\lambda > 0, \alpha < 0$ and $\beta < 0$ are constant and f(x) is a given function called the forcing term which is differentiable and has a compact support i.e. it is nonzero only in a closed bounded set. Equation (1) was first derived by Akylas (1984) and in an asymptotically reduced result from Euler equations of fluid motion and the corresponding boundary conditions. The unknown $\eta(x,t)$ represents the first order elevation of the free surface of the fluid from it is equilibrium level. The forcing function f(x) is due to the bottom topography of the fluid domain (such as a bump or dent) or due to an external pressure on the free surface such as the wind stress on the surface of an ocean. In the absence of the forcing function term i.e. f(x) = 0 equation (1) becomes the familiar Korteweg deVries(KdV) equation

$$\eta_t + \lambda\eta_x + 2\alpha\eta\eta_x + \beta\eta_{xxx} = 0 \qquad -\infty < x < \infty \quad t > 0 \quad (2)$$

when $\eta_t = 0$ equation (1) becomes the equation of our concern, the stationary forced (sfKdV) equation,

$$\lambda\eta_x + 2\alpha\eta\eta_x + \beta\eta_{xxx} = f_x(x) \qquad -\infty < x < \infty \quad (3)$$

Equation (3) was first derived by Shen (1993). The solutions of equation (3) are categorized according to the value of λ (Upstream flow velocity) as follows (Shen, 1993).

(i) Supercritical stationary waves. They occur only when λ is positive and sufficiently large.
(ii) Subcritical stationary waves. They occur only when λ is negative and sufficiently small.
(iii) Unsteady periodic soliton radiation. This solution appears when $|\lambda|$ is small. Such a solution is called the transcritical solution.

II. PROBLEM STATEMENT

In this paper we study supercritical solitary wave solutions of equation (3) with positive forcing terms in the rectangular bump. In this work we shall take the value of λ equal to 3 and the length of rectangular bump equals 2 in order to compare with the previous results.
Solitary waves refer to any surface wave profile that dies out at infinity which means that the free surface elevation η has the property

$$\eta \to 0, \quad \eta_x \to 0 \text{ and } \quad \eta_{xx} \to 0 \quad as \quad x \to \pm\infty \quad (4)$$

Integrating equation (3) once with respect to the independent variable from -∞ to x, we have

$$\lambda\eta + \alpha\eta^2 + \beta\eta_{xx} = f(x), \qquad -\infty < x < \infty, \quad \lambda > 0 \quad (5)$$

with $\eta \to 0, \quad and \quad \eta_x \to 0 \quad as \quad x \to \pm\infty$

In this paper we shall study equation (5) and (6) with the following conditions

(i) $\alpha = -\dfrac{3}{4} < 0, \quad \beta = -\dfrac{1}{6} \quad and\ \lambda = 3.$

(ii) $f(x) = \begin{cases} 1, & -\dfrac{a}{2} \le x \le \dfrac{a}{2} \\ 0 & otherwise \end{cases}$

where **a** is positive constant representing the length of a rectangular bump.

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In order to obtain the solution of equation (5) and (6) the domain of the problem has been divided in three parts. The left ($x \le -a/2$), the middle ($-a/2 < x < a/2$) and the right parts ($x \ge a/2$). The complete solution was found by matching the solutions of all parts. Section 3 and Section 4 describes the method of finding solution on the left and the right part respectively. Section 5 describes the solution in the middle part. In Section 6 we determine the phase shift L_o which depends on the forcing terms and the continuity conditions between each adjacent part. In Section 7 we combine the solutions obtained in Section 4 and Section 5. The conclusions of this paper are summarized in Section 8.

III. LEFT-SIDE SOLUTION

In this Section we need to solve equations (5) and (6) in the region of zero forcing term by using analytical method. Meanwhile, the solution in the region of non zero forcing term will be discussed later in Section 5.

Outside the region of the forcing term, equation (5) and (6) becomes

$$\lambda\eta + \alpha\eta^2 + \beta\eta_{xx} = 0 \qquad \frac{a}{2} \le x \le -\frac{a}{2}, \quad \lambda > 0 \qquad (7)$$

$$\eta \to 0, \quad and \quad \eta_x \to 0 \to 0 \quad as \quad x \to \pm\infty \qquad (8)$$

We need to find the solution in the negative region ($x \le -a/2$) outside the rectangular bump. In this region equations (7) and (8) are written as

$$\lambda\eta + \alpha\eta^2 + \beta\eta^{//} = 0 \qquad -\infty < x < -\frac{a}{2}, \quad \lambda > 0 \quad (9)$$

$$\eta \to 0, \quad and \quad \eta_x \to 0 \to 0 \quad as \quad x \to -\infty \qquad (10)$$

where equation (10) is the half negative solitary wave conditions. We start by rearranging equation (9) in the form

$$\beta\eta^{//} = -\lambda\eta - \alpha\eta^2 \qquad -\infty < x \le -\frac{a}{2}, \quad \lambda > 0 \qquad (11)$$

where $\eta^/ = \dfrac{d\eta}{dx}$

The procedure we adopt here in order to integrate equation (11) is that we write $\eta^{//}$ as

$$\eta^{//} = \eta^/ \frac{d\eta^/}{d\eta} \qquad (12)$$

Substituting equation (12) into equation (11) we obtain

$$\beta\eta^/ d\eta^/ = \left(-\lambda\eta - \alpha\eta^2\right)d\eta \qquad (13)$$

Equation (13) is simply a first order separable equation which can be integrated easily to get

$$\frac{\beta}{2}\left(\eta^/\right)^2 = -\frac{\lambda}{2}\eta^2 - \frac{\alpha}{3}\eta^3 + C \qquad (14)$$

Multiplying both sides of equation(14) by $\dfrac{3}{\alpha}$ we then obtain

$$\frac{3\beta}{2\alpha}\left(\eta^/\right) = -\frac{3\lambda}{2\alpha}\eta^2 - \eta^3 + \frac{3}{\alpha}C \qquad (15)$$

where $\dfrac{\beta}{\alpha}$ is positive constant and **C** is the integration constant.

By the half negative solitary wave conditions (10), equation (15) has a real double root, r_o and r_o is smaller than the third real root r. Then equation (15) can be written as

$$\frac{3\beta}{2\alpha}\left(\eta^/\right)^2 = (r-\eta)(\eta - r_o), \; r_o < r \qquad (16)$$

If we now let

$$\nu = \eta - r_o \qquad \zeta = \sqrt{\frac{2\alpha}{9\beta}}x,$$

then equation (16) can be written as

$$\frac{1}{3}\left(\frac{d\nu}{d\zeta}\right) = \nu^2(s_1 - \nu), \quad where \quad s_1 = r - r_o > 0. \qquad (17)$$

Equation (17) can be integrated to yield

$$\nu = s_1 \sec h^2 \sqrt{\frac{3s_1}{4}}\zeta \qquad (18)$$

Hence, the actual solution on the left side is

$$\eta(x) = \quad -\frac{3\lambda}{2\alpha}\sec h^2 \sqrt{\frac{-\lambda}{4\beta}}(x - L_o), \qquad (19)$$

where L_o is the phase shift to be determined in Section 6. All graphs of these solutions are shown in Figure 1. In Section 6 we shall show how to calculate L_o. For the purpose of illustration we now use the values of L_o obtained in Section 6 to produce the appropriate solution of the sfKdV on the left side.

The graphs shown in Figure 1 are the solution of the sfKdV equation on the left side when λ = 3 and a = 2. The values of L_o used for this purpose are -0.20752, -0.797065, -1.792481 and -2.248243 for graph (a), (b), (c) and (d) respectively.

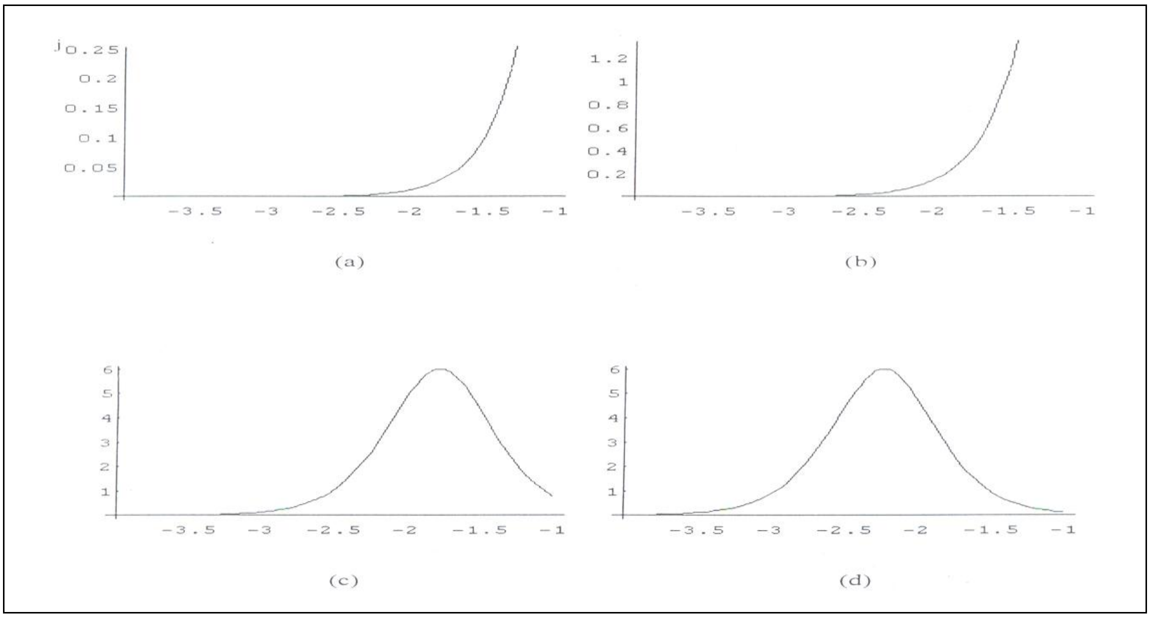

Figure 1: Left-Side Solution

IV. THE RIGHT-SIDE SOLUTION

In this section we determine the solution on the right side ($x \geq \frac{a}{2}$) that out side the rectangular bump. In this region equations (7) and (8) become

$$\lambda\eta + \alpha\eta^2 + \beta\eta'' = 0, \frac{a}{2} \le x < \infty, \lambda > 0 \qquad (20)$$

$$\eta \to 0, \eta_x \to 0 \qquad as \quad x \to +\infty \qquad (21)$$

where equation (21) in normally termed as the positive half solitary wave conditions.

The different between the left and the right region is simply on the solitary wave conditions. The same procedure as used for the left side can be applied to find the solution in the right and we thus obtain

$$\eta(x) = -\frac{3\lambda}{2\alpha}\sec h^2\sqrt{\frac{-\lambda}{4\beta}}(x - L_1), \qquad (22)$$

where the phase shift $L_1 = -L_o$. All graphs of this solution are shown in Figure 2.

The graphs shown in Figure 2 are the solution of the sfKdV equation on the right side when λ = 3 and a = 2. The values of L_1 used for this purpose are 0.20752, 0.797065, 1.792481 and 2.248243 for graph (a), (b), (c) and (d) respectively.

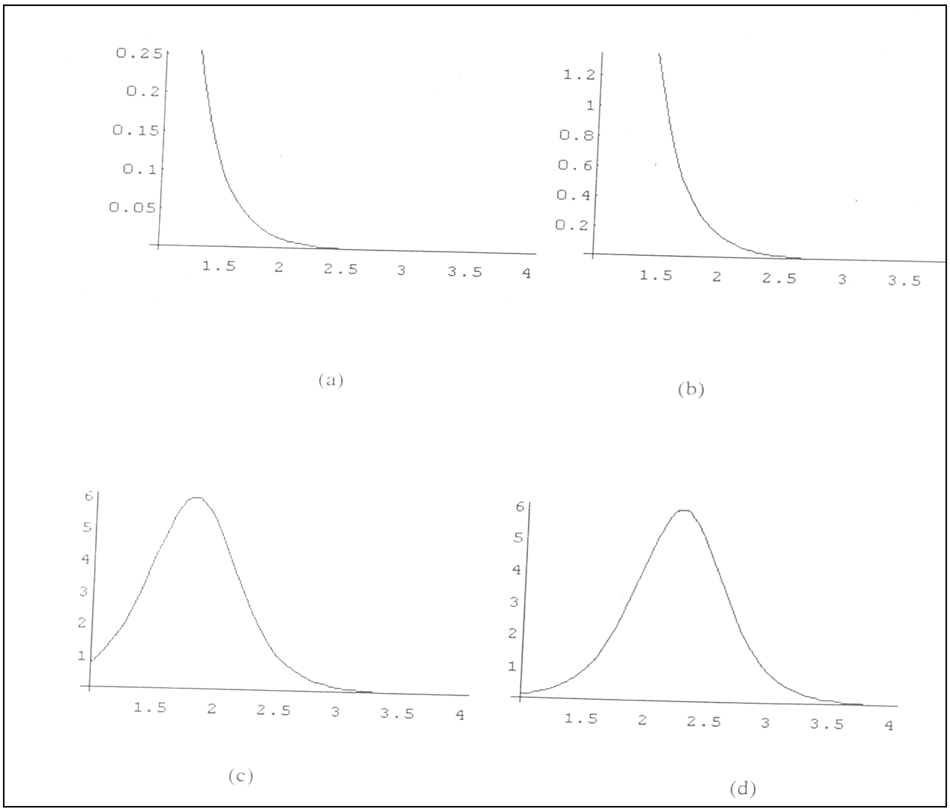

Figure 2: Right-Side Solution

V. The Middle Solution

At this stage, we have obtained the solutions at both sides outside the region of the forcing terms discussed in Section 3 and Section 4, respectively. What is left now is to find the solutions in the middle. This means that we need to solve equations (5) and (6) in the region of the forcing terms. We will show that the solution in this region can expressed in terms of Weierstrass elliptic functions. The solutions we obtained in Section 3, 4, and 5 will be combined to give the full solutions. This will be presented in Section 7.

In this region, $|x| \le \frac{a}{2}$, the solution must satisfy

$$\lambda\eta + \alpha\eta^2 + \beta\eta_{xx} = 1, \qquad \lambda > 0 \qquad (23)$$

We shall be dealing with equation (23). In order to solve this equation we need to deduce some conditions. These conditions come from continuity of η *and* $\eta^{/}$ *at* $x = -\frac{a}{2}$ *and* $x = \frac{a}{2}$.

Now at $x = -\frac{a}{2}$, we have

$$\eta\left(-\frac{a}{2}\right) \equiv \eta_o \qquad and \qquad \eta^{/}\left(-\frac{a}{2}\right) \equiv \eta_1 \tag{24}$$

where

$$\eta_o = -\frac{3\lambda}{2\alpha}\sec h^2\sqrt{\frac{-\lambda}{4\beta}}(x+L_o), \tag{25}$$

$$\eta_1 = \sqrt{-\frac{\lambda}{\beta}\eta_o}\tanh\sqrt{\frac{-\lambda}{4\beta}}\left(\frac{a}{2}+L_o\right), \tag{26}$$

At $x = \frac{a}{2}$ we have

$$\eta\left(-\frac{a}{2}\right) = \eta\left(\frac{a}{2}\right),$$
$$\eta^{/}\left(-\frac{a}{2}\right) = \eta^{/}\left(\frac{a}{2}\right) \quad or \quad \eta^{/}\left(-\frac{a}{2}\right) = -\eta^{/}\left(\frac{a}{2}\right) \tag{26*}$$

Integrating equation (23) from –a/2 to x (< a/2), we find

$$\frac{\beta}{2}\left(\eta^{/}\right)^2 = \eta - \frac{\lambda}{2}\eta^2 - \frac{\alpha}{3}\eta^3 + \eta_o \tag{27}$$

Equation (27) can be written as

$$\left(\eta^{/}\right)^2 = b_1\eta^3 + b_2\eta^2 + b_3\eta + b_4 \tag{28}$$

where

$$b_1 = -\frac{2\alpha}{3\beta}, \quad b_2 = -\frac{\lambda}{\beta}, \quad b_3 = \frac{2}{\beta} \quad and\ b_4 = b_3\eta_o$$

By making the transformation

$$\eta = c_1u + c_2$$

Equation (28) is converted into

$$c_1^{\,2}u^{/2} = b_1(c_1u+c_2)^3 + b_2(c_1u+c_2)^2 + b_3(c_1u+c_2) + b_4 \tag{29}$$

Dividing both sides of equation (29) by $(c_1)^2$ we obtain

$$(u')^2 = (b_1c_1)u^3 + \left(3b_1\left(-\frac{b_2}{3b_1}\right) + b_2\right)u^2 + \left(3b_1\left(\frac{b_2^{\,2}}{9b_1^{\,2}}\right) + 2b_2\left(-\frac{b_2}{3b_1}\right) + b_3\right)\frac{u}{c_1} +$$
$$\left(b_1c_2^{\,3} + b_2c_2^{\,2} + b_3c_2 + b_4\right)\frac{1}{c_1} \qquad (30)$$

Equation (30) is now written in the a simplified form as

$$u'^2 = 4u^3 - g_2u - g_3 \qquad (31)$$

where

$$c_1 = \frac{4}{b_1},\ c_2 = -\frac{b_2}{3b_1},\ g_2 = -\frac{(b_2c_2 + b_3)}{c_1} \quad and \quad g_3 = -\frac{\left(b_1c_2^{\,2} + b_3c_2 + b_4\right)}{c_1^{\,2}}$$

In equation (31), g_2 is constant and g_3 is a function of L_o for given λ, α, β and **a**. The general solution of equation (31) can be expressed in term of Weierstrass elliptic function (Gong, 1994)

$$u(x) = \wp(x + T, g_2, g_3)$$

where T is a constant.

Thus, in the region of the non-zero forcing term ($|x| \le \frac{a}{2}$) the supercritical solitary wave solution can be written as

$$\eta(x) = c_1\wp(x + T, g_2, g_3) + c_2 \qquad (32)$$

Equations (25) and (26) are now respectively reduced to

$$\eta_o = c_1\wp(-\frac{a}{2} + T, g_2, g_3) + c_2 \qquad (33)$$

and

$$\eta_1 = c_1\wp'(-\frac{a}{2} + T, g_2, g_3) \qquad (34)$$

From Gong and Shen (1993), we find the identity

$$(x + T, g_2, g_3) = \frac{1}{4}\left\{\frac{\wp'(x, g_2, g_3) - \wp'(T, g_2, g_3)}{\wp(x, g_2, g_3) - \wp(T, g_2, g_3)}\right\}^2 - \wp(x, g_2, g_3) - \wp(T, g_2, g_3) \qquad (35)$$

Substituting equations (33) and (34) into equation (35), we then obtain

$$(x + T, g_2, g_3) = \frac{1}{4}\left\{\frac{\wp'(x, g_2, g_3) - \frac{\eta_1}{c_1}}{\wp(x, g_2, g_3) - (\frac{\eta_o - c_2}{c_1})}\right\}^2 - \wp(x, g_2, g_3) - \frac{\eta_0 - c_2}{c_1} \qquad (36)$$

We now observe that in equation (36), the right hand side is independent of constant T. We then have

$$(x+T,g_2,g_3)=\frac{1}{4}\left\{\frac{c_1\wp'(x,g_2,g_3)-\eta_1}{\wp(x,g_2,g_3)-\eta_o+c_2}\right\}^2-\wp(x,g_2,g_3)-\frac{\eta_0-c_2}{c_1} \qquad (37)$$

Using Mathematica program Substituting equation(37) into equation (32), we get the solution on the middle side. All graphs of these solutions are shown in Figure 3.

Figure 3 shows the solutions of the sfKdV in the middle, when λ = 3 and a = 2. The values of L_o are -0.20752, -0.792481, -1.792481 and -2.248243 for graph (a), (b), (c) and (d) respectively.

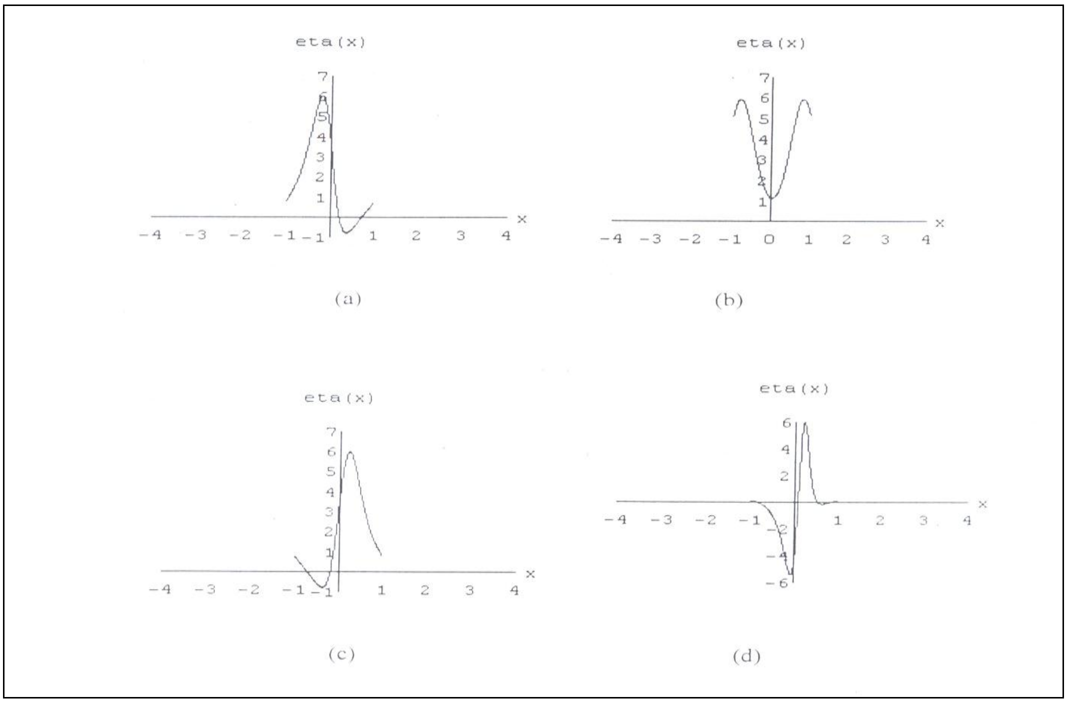

Figure 3: The Middle Solution

VI. The Phase Shift L_o

In this section we shall determine the phase shift L_o which depends on the forcing terms and the continuity conditions between each adjacent part.

Substituting x = a/2 and x = -a/2 into equation (32) we get

$$\eta\left(\frac{a}{2}\right)=c_1\wp\left(\frac{a}{2}+T,g_2,g_3\right)+c_2 \qquad and \qquad \eta\left(-\frac{a}{2}+T,g_2+g_3\right)+c_2.$$

Substituting the above expression in equation (26*) we obtain

$$\wp\left(\frac{a}{2}+T,g_2,g_3\right)-\wp\left(-\frac{a}{2}+T,g_2+g_3\right)=0 \qquad (38)$$

From equations (33), (34), (37) and (38) we can obtain

$$B(\lambda, L_o) \equiv \frac{c_1}{4}\left\{\frac{c_1\wp'(a, g_2, g_3) - \eta_1}{c_1\wp(a, g_2, g_3) - \eta_o + c_2}\right\}^2 - c_1\wp(a, g_2, g_3) + 2c_2 - 2\eta_o = 0 \quad (39)$$

Equation (39) determines the phase shift L_o.

Equation (39) have a solitary wave solution only when $B(\lambda, L_o) = 0$ (Gong,1994). Hence, $B(\lambda, L_o) = 0$ is the condition used to calculate the phase shift L_o. By using Mathematica program the function $B(\lambda, L_o)$ was plotted versus trial L_o. The intersection point with L_o -axis was found as the values of L_o .(see Figure 4)

This curve has four intersection point with L_o -axis, i.e. $B(\lambda, L_o)$ has four zero, $L_0 = -0.20752$, $-0.797065, -1.792481$ and -2.248243 that means we have four supercritical solitary wave solutions.

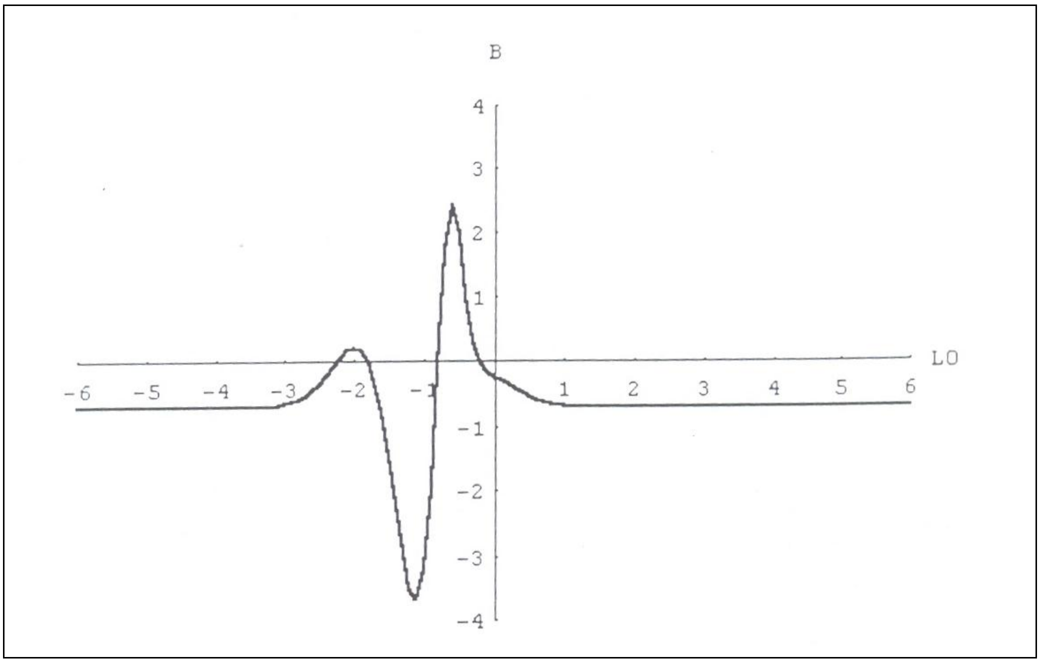

Figure 4: The Curve of $B(\lambda, L_0)$ versus L_0

VII. Complete Solutions

We have obtained the solutions of the sfKdV equation at both sides and also in the middle as discussed in Section 3, 4, and 5, respectively. In this Section we shall give the complete solutions of the sfKdV equation. This means that we need to combine the solutions in both side with the middle ones.

We rewrite the solutions outlined in Section 3 as

$$\eta(x) = -\frac{3\lambda}{2\alpha}\sec h^2 \sqrt{\frac{-\lambda}{4\beta}}(x - L_o), \qquad (40)$$

for the left side, and

$$\eta(x) = -\frac{3\lambda}{2\alpha}\sec h^2 \sqrt{\frac{-\lambda}{4\beta}}(x - L_1), \qquad (41)$$

for the right side.

The solution in the middle parts

$$\eta(x) = c_1 \wp(x+T, g_2, g_3) + c_2 \qquad (42).$$

By matching equations (40), (41) and (42) we get

$$\eta(x) = \begin{cases} -\dfrac{3\lambda}{2\alpha}\sec h^2\sqrt{-\dfrac{\lambda}{4\beta}}(x-L_o), & -\infty < x \le -\dfrac{a}{2}; \\ c_1\wp(x+T, g_2, g_3) + c_2 & -\dfrac{a}{2} < x < \dfrac{a}{2}; \\ -\dfrac{3\lambda}{2\alpha}\sec h^2\sqrt{-\dfrac{\lambda}{4\beta}}(x-L_1), & \dfrac{a}{2} \le x < \infty. \end{cases} \qquad (43)$$

Equation (43) is the complete solution of sfKdV equation.

The graphs shown in Figure 5 are the complete solutions of the sKdV equation when $\lambda = 3$ and a = 2. The values of L_o are -0.20752, -0.797065, -1.792481 and -2.248243 for graph (a), (b), (c) and (d) respectively

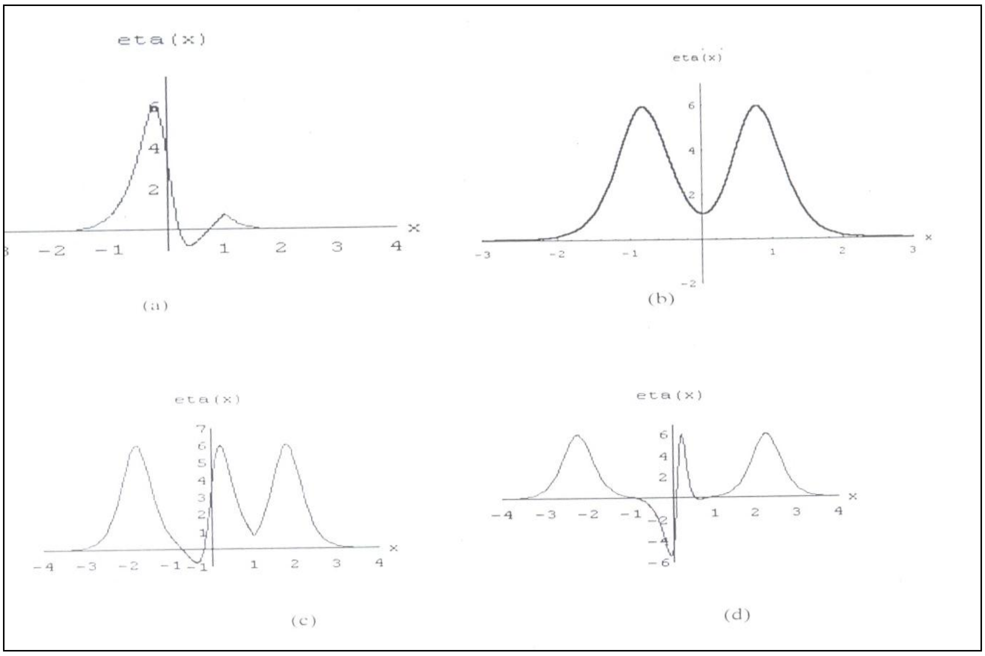

Figure 5: The Complete Solution

VIII. CONCLUSION

In this paper we have studied the solutions of the stationary Korteweg-deVries (sfKdV) equation. In order to obtain the solution of the sfKdV equation, the domain of the problem has been divided into three parts, the left, the middle and the right parts. The solutions on the left and right parts were obtained in Section 3 and Section 4 by using analytical method. The solution on the middle were obtained in Section 5. In both cases we have constructed two computer programs by using Mathematica to produce the final solutions. The complete solution of the sfKdv equation is given in Section 7, by matching the solutions of the three parts i.e. the left, the middle and the right parts. We have found that the solutions of the stationary Korteweg-deVries (sfKdV) equation produce four different solutions according to the value of the phase shift L_o. Only one solution is positive the three others solutions have positive and negative values.. Further research can be carried out for negative forcing terms

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