Dynamic Response Comparison for Symmetric Beams using 3-noded and 2-noded Beam Element

Mohammad Anas, M. Naushad Alam

***Abstract*— An efficient one dimensional 3-noded finite element model has been developed for the vibrational analysis of composite beam for various boundary conditions, using the efficient layerwise zigzag theory. The results are compared with 2-noded beam element model. To meet the convergence requirements for the weak integral formulation, fifth power Hermite interpolation is used for the transverse displacement and quadratic interpolation is used for the axial displacement and shear rotation. Each node of an element has four degrees of freedom. The formulation is validated by comparing the results of the 2D finite element (2D-FE) for the simply supported beam. The present zigzag finite element results for natural frequencies and mode shapes of the beam are obtained with one-dimensional finite element (1DFE) codes developed in MATLAB. This comparison establishes the accuracy of zigzag finite element analysis for natural frequencies of the symmetric laminated beams, and helps to obtain natural frequencies of fixed beam and cantilever beam with less error.**

***Index Terms*— composite laminates, sandwich beam, Vibration analysis, zigzag theory, FEM, MATLAB.**

I. INTRODUCTION

Composite structures are increasingly used in areas like automotive engineering and other applications as they posses lower weight and higher strength and stiffness than those composed of other metallic materials. For design of composite and sandwich beams accurate knowledge of deflection and stress under static and dynamic loadings, natural frequencies, mode shapes are required. Exact elasticity solutions [1-3] have been provided for static, free and forced vibration cases. Discrete layer theories with layer wise displacement approximation are quite accurate but computationally expensive as the number of basic variables depends on number of layers.

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Among most discrete layer theories [4, 5] the shear stress continuity at layer interfaces, is violated. Kapuria et. al. [6, 11] presented an assessment of zigzag theory for laminated composite beams by giving analytical solution for simply supported end conditions only Benjeddou [7] has presented finite element modeling of adaptive structures. Kapuria et. al. [8] presented a novel finite element model of efficient zigzag theory for static analysis of hybrid piezoelectric beams. They presented finite element analysis of hybrid piezoelectric beams under static electromechanical load using zigzag theory. They also compared then results with 2D finite element results obtained using ABAQUS to establish the accuracy zigzag theory. Navier type solutions for simply supported beams were presented in refs. [6, 9] which did not provide finite element formulation of zigzag theory. One of the authors [10] recently presented efficient layer wise finite element model for dynamic analysis of laminated piezoelectric beams. Two noded beam element model developed by Alam et. al [13] is an initiative approach in the field of F E analysis.

This work considers a three nodded finite element model for dynamic analysis for composite beams based on zigzag theory 6 in which the shear traction condition at the top and bottom and the transverse shear continuity condition at the layer interfaces are satisfied. 5^{th} Hermite interpolation function [12] is used for deflection and quadratic interpolation is used for the axial displacement and rotation. The finite element formulation and the MATLAB code developed thereof are validated by comparison of the results with the results obtained using ABAQUS software. The one-dimensional finite element (1D-FE) results for propped beam are compared with 2D finite element (2D-FE) results.

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II. FINITE ELEMENT MODEL USING 3-NODED BEAM ELEMENT

With reference to two noded element model [13], three nodded elements are used for the displacement variables. The primary variables, u_0 w_0, ψ_0, within an element are expressed in terms of their nodal values using appropriate polynomial interpolation functions. The highest derivatives of u_0, w_0, ψ_0,, appearing in the Variational equation [13] are $u_{0,x}$, $w_{0,xx}$, $\psi_{0,x}$. To meet the convergence requirements of the finite element method, u_0, $w_{0,x}$, ψ_0, must be continuous at the element boundaries. Hence w_0 is expanded using 5^{th} power interpolation along x in terms of the nodal values of w_0, and $w_{0,x}$. Similarly, quadratic Lagrange interpolation along x is used for u_0 and ψ_0 in terms of their nodal values. Thus at the element level, each node will have four degrees of freedom u_0, w_0, $w_{0,x}$, ψ_0, for the displacements. The values of an entity (…) at the nodes 1, 2 and 3 are donated by $(\ldots)_1$, $(\ldots)_2$, and $(\ldots)_3$ respectively.

The following interpolations of u_0, w_0, ψ_0, have been used in terms of the nodal values and the shape function matrices N *and* $\overline{N}$:

$$u = N.\, u_0^e\ ,\quad \psi_{0,} = N.\psi_0^e,\quad w_0 = \overline{N}\, w_0^e. \tag{10}$$

with
$$u_0^e = \begin{bmatrix} u_{01} \\ u_{02} \\ u_{03} \end{bmatrix} \quad \psi_0^e = \begin{bmatrix} \psi_{01} \\ \psi_{02} \\ \psi_{03} \end{bmatrix}, \qquad w_0^e = \begin{bmatrix} w_{01} \\ w_{0,x1} \\ w_{02} \\ w_{0,x2} \\ w_{03} \\ w_{0,x3} \end{bmatrix},$$

$$N = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix}, \tag{11}$$

$$\overline{N} = \begin{bmatrix} \overline{N}_1 & \overline{N}_2 & \overline{N}_3 & \overline{N}_4 & \overline{N}_5 & \overline{N}_6 \end{bmatrix}$$

$N_1=2\,x^2/L_e^2-x/L_e$, $\quad N_2=1-4(x/L_e)^2$,
$N_3= x/L_e+2x^2/L_e^2$

Similarly, other shape functions are derived using interpolation [12] as:

$\overline{N}_1= x^2/L_e^2 -5/4.x^3/L_e^3-1/2.x^4/L_e^4+3/4.x^5/L_e^5$

$\overline{N}_2= 1/4\,.x^2/L_e +1/4.\,x^3/L_e^2-1/4.x^4/L_e^3+1/4.x^5/L_e^4$

(12)

$\overline{N}_3= 1-2x^2/L_e^2 + x^4/L_e^4$,

$\overline{N}_4=x -2x^3/L_e^2+x^5/L_e^4$

$\overline{N}_5= x^2/L_e^2 +5/4.x^3/L_e^3-1/2.x^4/L_e^4-3/5.x^5/L_e^5$

$\overline{N}_6= -1/4.x^2/L_e -1/4.\,x^3/L_e^2+1/4.x^4/L_e^3+1/4.x^5/L_e^4$

The integrand in the variational equation can be expressed as, [ref. 13, equaion no. 23]

$$T^e = \int_0^{Le} \delta U^{eT}\,[B_m^T.\hat{I}B_m\ddot{U}^e + \hat{B}^T\hat{D}.\hat{B}U_e - B_{m2}^T F_2]\delta x \quad =0$$

(13)

N_x, M_x, P_x, *and* Q_x are substituted from ref. [13] to obtain general equation after integration as:

$$M^e\ddot{U}^e + K^eU^e = P^e \tag{14}$$

where $U^e = [u_0^{e\,T}\quad w_0^{e\,T}\quad \psi_0^e]^T$, $M^e = \int_{-Le}^{Le} B_m^T\hat{I}B_m dx.$,

$P^e = [\,0\quad \overline{N}^TF_2\quad 0]$, $\quad K^e = \int_{-Le}^{Le} B^T\hat{I}B\; dx.$

The natural frequency for free vibration can be obtained from the above equation by making it aneigen value problem as

$$K^eU^e - \omega_n^2\, M^eU^e = 0 \tag{15}$$

The beam generalized displacement are related to displacement vector

U^e *(displacement vector)* $= [u_0^{e\,T}\quad w_0^{e\,T}\quad \psi_0^e]^T$

that is $\quad \bar{u}_1 = B_{m1}U^e \quad$ and $\quad \bar{u}_2 = B_{m2}U^e$

$$B_{m1} = \begin{bmatrix} N & 0 & 0 \\ & -\overline{N}_{,x} & 0 \\ 0 & & N \end{bmatrix} \quad and \quad B_m = \begin{bmatrix} B_{m1} \\ B_{m2} \end{bmatrix} = \begin{bmatrix} N & 0 & 0 \\ 0 & -\overline{N}_{,x} & 0 \\ 0 & 0 & N \\ 0 & \overline{N} & 0 \end{bmatrix}$$

Refer appendix for the equations and matrices.

III. RESULT AND DISCUSSION

A highly inhomogeneous symmetric beam is analyzed for simply supported boundary condition. The stacking order is mentioned from the bottom. The beams is composite beam of material [11] with $Y_1 = 181GPa$ *and* $Y_2 = Y_3 = 10.3$ *GPa and* $\nu_{12} = \nu_{13} = .25$, $\nu_{23} = .33$ .consisting of four plies of equal thickness 0.25h. It has symmetric lay-up [0/90/90/0]. The density of materials of the beam is 1578 kg/m^3 [11].Flexural natural frequencies and mode shapes of laminated composite beams have been computed by developing 1D-FE MATLAB program [11]. [Table 1]

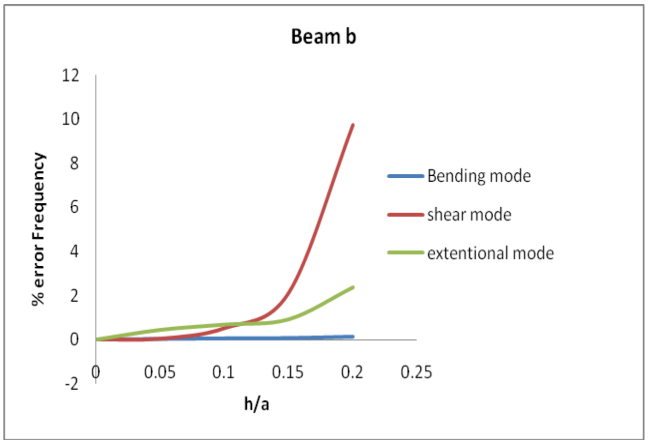

Fig. 2 % error in $\overline{\omega}_1$ for 1st longitudinal mode using 2 noded beam elements

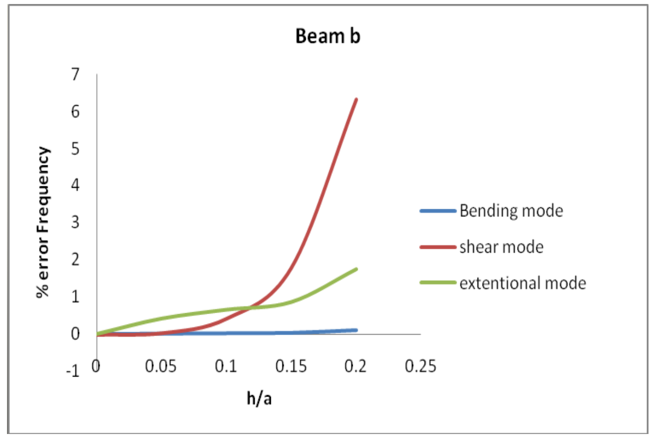

Fig. 3 % error in $\overline{\omega}_1$ for 1st longitudinal mode using 3-noded beam elements

Table 1 % error of natural frequencies of symmetric beam (b) simply supported boundary condition using FEM and 2D exact values

n	m	S	2D exact (Availbl)	% error 2-noded	% error 3-noded
1	1	5	6.8060	0.13	0.1
	1	10	9.3434	0.068	0.04
	1	20	10.640	0.056	0.03
	1	100	11.193	0.032	0.02
2	1	5	16.515	1.33	.91
	1	10	27.224	.14	0.08
	1	20	37.374	.047	0.02
	1	100	44.477	0.02	.011
3	1	5	26.688	4.41	2.81
	1	10	46.419	0.54	0.31
	1	20	71.744	0.022	0.02
	1	100	98.988	0.001	0.001

IV. CONCLUSION

The present FE model is developed for vibration analysis of symmetric beam for various end conditions. The presented model can be used for computing frequencies for various end conditions. The comparison shows that the 1DFE model of zigzag theory yields very accurate results for mode shapes for simply supported beams.

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V. APPENDIX

M^e *(Inetia matrix)* = $\int_{-Le}^{Le} B_m^T.\hat{I}.B_m dx = M_1 + M_2$

$$M_1 = \begin{bmatrix} I_{11}c_{11} & -I_{12}c_{12} & I_{13}c_{11} \\ & I_{22}c_{13} & -I_{23}c_{12}^T \\ sym & & I_{33}c_{11} \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 0 & 0 \\ & \bar{I}_{22}c_{14} & 0 \\ sym & & 0 \end{bmatrix}, \quad c_{11} = \int_{-Le}^{L_e} N^T.N dx,$$

$$c_{12} = \int_{-Le}^{L_e} N^T.\bar{N}_{,x} dx, \quad c_{13} = \int_{-Le}^{L_e} \bar{N}_{,x}^T.\bar{N}_{,x} dx, \quad c_{14} = \int_{-Le}^{L_e} \bar{N}^T.\bar{N} dx$$

K^e *(stiffness matrix)*= $\int_{-Le}^{Le} \hat{B}^T.\hat{D}.\hat{B} dx = K_1 + K_2$, where

$$K_1 = \begin{bmatrix} A_{11}c_8 & -A_{22}c_9 & A_{13}c_8 \\ & A_{22}c_{10} & -A_{23}c_9^T \\ sym & & A_{33}c_8 \end{bmatrix} \text{ and } K_2 = \begin{bmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ sym & & \bar{A}_{33}c_{11} \end{bmatrix}$$

the constants are given by

$$c_8 = \int_{-le}^{L_e} N_{,x}{}^T.N_{,x} dx, \quad c_9 = \int_{-Le}^{L_e} N_{,x}{}^T.\bar{N}_{,xx} dx, \quad c_{10} = \int_{-Le}^{L_e} \bar{N}_{,xx}^T.\bar{N}_{,xx} dx$$

c_8=
```
[  38/(3*Le), -64/(3*Le),  26/(3*Le)]
[ -64/(3*Le), 128/(3*Le), -64/(3*Le)]
[  26/(3*Le), -64/(3*Le),  38/(3*Le)]
```

c_9=
```
[ 4/Le^2,  13/Le, 0, 0, -4/Le^2,  3/Le]
[-8/Le^2, -24/Le, 0, 0,  8/Le^2, -8/Le]
[ 4/Le^2,  11/Le, 0, 0, -4/Le^2,  5/Le]
```

c_{10}=
```
[ 1273/(70*Le^3),  779/(70*Le^2), -64/(5*Le^3),  96/(7*Le^2),
[  779/(70*Le^2),    586/(35*Le), -32/(5*Le^2),     32/(7*Le),
[   -64/(5*Le^3),   -32/(5*Le^2), 128/(5*Le^3),             0,
[    96/(7*Le^2),      32/(7*Le),            0,    128/(7*Le),
[ -377/(70*Le^3), -331/(70*Le^2), -64/(5*Le^3), -96/(7*Le^2),
[  121/(70*Le^2),    124/(35*Le),  32/(5*Le^2),     32/(7*Le),

-377/(70*Le^3),  121/(70*Le^2)]
```

```
-331/(70*Le^2),    124/(35*Le)]
  -64/(5*Le^3),     32/(5*Le^2)]
  -96/(7*Le^2),       32/(7*Le)]
1273/(70*Le^3), -569/(70*Le^2)]
-569/(70*Le^2),    166/(35*Le)]
```

c_{11}=

```
[   (34*L_e)/15, -(28*L_e)/15,   (14*L_e)/15]
[ -(28*L_e)/15,   (46*L_e)/15, -(28*L_e)/15]
[   (14*L_e)/15, -(28*L_e)/15,   (34*L_e)/15]
```

c_{12}=

```
[ -146/105,    (29*L_e)/21,  16/15, -(64*L_e)/105,  34/105,       L_e /21]
[      5/7,  -(57*L_e)/35,       0, (128*L_e)/105,    -5/7,  -(8*L_e)/35]
[  -34/105, (131*L_e)/105, -16/15, -(64*L_e)/105, 146/105, (19*L_e)/105]
```

c_{13}=

```
[  139/(105*Le),       -61/105, -128/(105*Le),          8/21,  -11/(105*Le),        -1/70]
[       -61/105,  (731*Le)/630,        -8/105, -(152*Le)/315,         23/35, (22*Le)/315]
[ -128/(105*Le),        -8/105,  256/(105*Le),             0, -128/(105*Le),        8/105]
[          8/21, -(152*Le)/315,             0,  (256*Le)/315,         -8/21, -(8*Le)/315]
[  -11/(105*Le),         23/35, -128/(105*Le),         -8/21,  139/(105*Le),      -13/210]
[         -1/70,   (22*Le)/315,         8/105,   -(8*Le)/315,       -13/210,    (4*Le)/45]
```

c_{14}=

```
[  (1046*Le)/3465,  -(24*Le^2)/385,      (8*Le)/63,  -(32*Le^2)/693,
[  -(24*Le^2)/385, (139*Le^3)/2310,  (8*Le^2)/315,  (64*Le^3)/3465,
[       (8*Le)/63,    (8*Le^2)/315,  (256*Le)/315,               0,
[  -(32*Le^2)/693,  (64*Le^3)/3465,              0, (256*Le^3)/3465,
[   (131*Le)/3465, (359*Le^2)/3465,      (8*Le)/63,   (32*Le^2)/693,
[ -(29*Le^2)/3465, -(67*Le^3)/6930, -(8*Le^2)/315,  -(8*Le^3)/1155,
```

```
 (131*Le)/3465,   -(29*Le^2)/3465]
(359*Le^2)/3465, -(67*Le^3)/6930]
     (8*Le)/63,   -(8*Le^2)/315]
  (32*Le^2)/693,  -(8*Le^3)/1155]
 (1046*Le)/3465, -(38*Le^2)/1155]
-(38*Le^2)/1155, (16*Le^3)/3465]
```