Ten Problems of Number Theory

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Abstract— **In this paper we will show solution of ten problems in number theory.**

Problem 1 : Balanced Primes are infinite.

Problem 2 : Euclid primes are infinite.

Problem 3 : Quasiperfect number doesn't exist.

Problem 4 : Prime triplets are infinite.

Problem 5 : Odd Superperfect number doesn't exist.

PROBLEM 6 : PROOF OF POLIGNAC'S CONJECTURE.

PROBLEM 7 : THERE ARE INFINITELY MANY PRIME OF THE FORM N²+1.

PROBLEM 8 : LONELY RUNNER CONJECTURE PROOF WHEN VELOCITIES ARE IN ARITHMETIC PROGRESSION.

PROBLEM 9 : WE WILL FIND THE VALUE OF IMAGINARY PART OF NON-TRIVIAL ZEROS OF RIEMANN ZETA FUNCTION.

PROBLEM 10 : WE WILL PROVE LANDER, PARKIN AND SELFRIDGE CONJECTURE.

Index Terms— **Balanced Primes, Euclid primes, Prime triplets, Polignac's conjecture.**

I. Problem 1 :

A **balanced prime** is a prime number that is equal to the arithmetic mean of the nearest primes above and below. Or to put it algebraically, given a prime number p_n, where *n* is its index in the ordered set of prime numbers,

$$p_n = \frac{p_{n-1} + p_{n+1}}{2}.$$

The first few balanced primes are

5, 53, 157, 173, 211, 257, 263, 373, 563, 593, 607, 653, 733, 947, 977, 1103 (sequence A006562 in OEIS).

For example, 53 is the sixteenth prime. The fifteenth and seventeenth primes, 47 and 59, add up to 106, half of which is 53, thus 53 is a balanced prime.

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When 1 was considered a prime number, 2 would have correspondingly been considered the first balanced prime since

$$2 = \frac{1+3}{2}.$$

It is conjectured that there are infinitely many balanced primes.

Problem source : http://en.wikipedia.org/wiki/Balanced_prime

Solution

Let there are finite number of balanced primes.
We are considering primes of the form p-12, p-6, p.
Now let's p_n -6 be the last balanced prime.
Now we form odd numbers as below :
Red coloured numbers are divisible by 3. Blue coloured numbers are divisible by 5 and Green coloured numbers are divisible by 7.

p_n-10 p_n-8 p_n-6
p_n-4 p_n-2 p_n
p_n+2 p_n+4 p_n+6
p_n+8 p_n+10 p_n+12
p_n+14 p_n+16 p_n+18
p_n+20 p_n+22 p_n+24
p_n+26 p_n+28 p_n+30
p_n+32 p_n+34 p_n+36
p_n+38 p_n+40 p_n+42
p_n+44 p_n+46 p_n+48
p_n+50 p_n+52 p_n+54
p_n+56 p_n+58 p_n+60
p_n+62 p_n+64 p_n+66
p_n+68 p_n+70 p_n+72
p_n+74 p_n+76 p_n+78
p_n+80 p_n+82 p_n+84
p_n+86 p_n+88 p_n+90
p_n+92 p_n+94 p_n+96
p_n+98 p_n+100 p_n+102
p_n+104 p_n+106 p_n+108

We see that p_n+96 and p_n+102 are composite and forming a balanced prime sets viz. p_n+94, p_n+100 and p_n+106.
Now according to our assumption one of this must be divisible by any prime before it i.e. one of them have to be composite.
Now, we will find infinite number of balanced prime sets only by 5 and 7 and it will occur in a regular frequency i.e. after every 70 numbers and in a particular column it will occur after every 210 numbers.
There are only finite number of primes before p_n.

- ⇨ They cannot make every balanced prime set unbalanced by dividing any of three primes forming the balanced prime set.

So, there needs to be born primes after p_n so that every regular occurrence of balanced prime set by 5,7 can be made unbalanced by dividing with those primes at least one of the prime forming balanced set.
But there is no regular occurrence of prime whereas 5,7 will be forming balanced set of primes in a regular manner.

- ⇨ There will be a shortcoming somewhere of primes to make each and every balanced pair formed by 5,7 in a regular manner.
- ⇨ There will be balanced prime after p_n.

Here is the contradiction.

- ⇨ Balanced primes are infinite.

Proved.

II. Problem 2 :

In mathematics, **Euclid numbers** are integers of the form $E_n = p_n\# + 1$, where $p_n\#$ is the nth primorial, i.e. the product of the first n primes. They are named after the ancient Greek mathematician Euclid.
The first few Euclid numbers are 3, 7, 31, 211, 2311, 30031, 510511 (sequence A006862 in OEIS).

It is not known whether or not there are an infinite number of prime Euclid numbers.

$E_6 = 13\# + 1 = 30031 = 59 \times 509$ is the first composite Euclid number, demonstrating that not all Euclid numbers are prime.
A Euclid number cannot be a square. This is because Euclid numbers are always congruent to 3 mod 4.

For all $n \geq 3$ the last digit of E_n is 1, since $E_n - 1$ is divisible by 2 and 5.

Problem source : http://en.wikipedia.org/wiki/Euclid_number

Solution

Let Euclid prime is finite.

Let the last Euclid prime is $E_n = p_n\# + 1 = p_1\, p_2 \ldots\ldots p_n + 1$

Now, $E_n = p_n\# + 1 = p_1\, p_2 \ldots\ldots p_n + 1 \equiv 1 \pmod 3$ and $\equiv -1 \pmod 4$

- ⇨ $E_k = 12p + 7$ form.
- ⇨ $E_k \equiv 3 \pmod 8$ (considering p as odd)

Now there will be cases when $p_1\, p_2 \ldots\ldots p_k \equiv \pm 2 \pmod 8$

There will be cases when $p_1\, p_2 \ldots\ldots p_k \equiv -2 \pmod 8$

- ⇨ $p_1\, p_2 \ldots\ldots p_n + 1 \equiv 7 \pmod 8$ which is contradiction.
- ⇨ p can be even.

Putting 2p in place of p we get, $E_k = 24p + 7$ (p is odd)

Now $24p + 7 \equiv -1 \pmod{16}$

Now, $p_1\, p_2 \ldots\ldots p_k \equiv \pm 2, \pm 6 \pmod{16}$

- ⇨ $p_1\, p_2 \ldots\ldots p_k + 1 \equiv -1, 3, -5, 7 \pmod{16}$
- ⇨ There will be cases when $p_1\, p_2 \ldots\ldots p_k + 1$ is not $\equiv -1 \pmod{16}$ which is contradiction.
- ⇨ p is even.

In this way the problem will go increasing giving same solution.

- ⇨ There will be primes after E_n.
- ⇨ Euclid primes are infinite.

Proved.

III. Problem 3 :

In mathematics, a **quasiperfect number** is a theoretical natural number n for which the sum of all its divisors (the divisor function $\sigma(n)$) is equal to$2n + 1$. Quasiperfect numbers are abundant numbers.

No quasiperfect numbers have been found so far, but if a quasiperfect number exists, it must be an odd square number greater than 10^{35} and have at least seven distinct prime factors.

Problem source : http://en.wikipedia.org/wiki/Quasiperfect_number

Solution

No odd number can be Quasiperfect number as there are even number of odd factors in odd number. (except square number)
Now, all even number has at least one odd factor excluding one except the numbers of the form 2^n.
Now, let's say an even number has 2 odd factors : p_1 and p_2

- ⇨ There are 3 odd factors viz. p_1, p_2, p_1p_2
- ⇨ Including 1 the sum of the odd factors become even.
- ⇨ Even number who has even number of odd factors cannot be Quasiperfect number.

Now, let's say an even number has 3 odd factors viz. p_1 , p_2 , p_3

- ⇨ There are 7 odd factors viz. p_1, p_2, p_3 p_1p_2, p_2p_3, p_3p_1, $p_1p_2p_3$
- ⇨ Including 1 the sum of the odd factors become even.
- ⇨ Even number who has odd number of odd factors cannot be Quasiperfect number.

Now, even number of the form 2^n has the factor sum 1+2+2²+...2^n = 2^(n+1) – 1. Obviously this doesn't satisfy the condition of being a Quasiperfect number.

- ⇨ No even number can be Quasiperfect number.

Now, let's say N is odd perfect square number.
N = {(a^n)(b^m)...(k^y)}² = (a^2n)(b^2m)....(k^2y)
Now, 1+a+a²+...+a^2n ≡ n(1+a) + 1 (mod 4) as a² ≡ 1 (mod 4)
As a is odd (1+a) is even.

- ⇨ n(1+a) is not divisible by 4 otherwise it will give remainder as 1.

⇨ n is odd and (1+a) = 2i where i is odd.

Putting 2n+1 in place of n and 2a+1 in place of a the LHS of the equation becomes :

$\{1+(2a+1)+(2a+1)^2+....+(2a+1)^{(4n+2)}\}.....\{1+(2k+1)+(2k+1)^2+....+(2k+1)^{(4y+2)}\}$

Now, LHS ≡ {(1+2a+1)((2n+1) +1}......{(1+2k+1)(2y+1) + 1} (mod 8)

⇨ LHS ≡ {2(1+a)(2n+1)+1}......{2(1+k)(2y+1)+1} (mod 8)

⇨ LHS ≡ 5*5*....*5 (mod 8) as (1+a), ..., (1+k) are even.

⇨ LHS ≡ 5 or 1 (mod 8) (if there are odd number of terms then 5, if even then 1)

But, RHS ≡ 2*1 +1 = 3 (mod 8) (as any odd square number ≡ 1 (mod 8))

Here is the contradiction.

⇨ Quasiperfect number doesn't exist.

Proved.

IV. Problem 4 :

In mathematics, a **prime triplet** is a set of three prime numbers of the form (p, $p + 2$, $p + 6$) or (p, $p + 4$, $p + 6$). With the exceptions of (2, 3, 5) and (3, 5, 7), this is the closest possible grouping of three prime numbers, since every third odd number greater than 3 is divisible by 3, and hence not prime.

The first prime triplets (sequence A098420 in OEIS) are

(5, 7, 11), (7, 11, 13), (11, 13, 17), (13, 17, 19), (17, 19, 23), (37, 41, 43), (41, 43, 47), (67, 71, 73), (97, 101, 103), (101, 103, 107), (103, 107, 109), (107, 109, 113), (191, 193, 197), (193, 197, 199), (223, 227, 229), (227, 229, 233), (277, 281, 283), (307, 311, 313), (311, 313, 317), (347, 349, 353), (457, 461, 463), (461, 463, 467), (613, 617, 619), (641, 643, 647), (821, 823, 827), (823, 827, 829), (853, 857, 859), (857, 859, 863), (877, 881, 883), (881, 883, 887)

A prime triplet contains a pair of twin primes (p and $p + 2$, or $p + 4$ and $p + 6$), a pair of cousin primes (p and $p + 4$, or $p + 2$ and $p + 6$), and a pair of sexy primes (p and $p + 6$).

A prime can be a member of up to three prime triplets - for example, 103 is a member of (97, 101, 103), (101, 103, 107) and (103, 107, 109). When this happens, the five involved primes form a prime quintuplet.

A prime quadruplet (p, $p + 2$, $p + 6$, $p + 8$) contains two overlapping prime triplets, (p, $p + 2$, $p + 6$) and ($p + 2$, $p + 6$, $p + 8$).

Similarly to the twin prime conjecture, it is conjectured that there are infinitely many prime triplets. The first known gigantic prime triplet was found in 2008 by Norman Luhn and François Morain. The primes are (p, $p + 2$, $p + 6$) with $p = 2072644824759 \times 2^{33333} - 1$. As of May 2013 the largest known prime triplet contains primes with 16737 digits and was found by Peter Kaiser. The primes are (p, $p + 4$, $p + 6$) with $p = 6521953289619 \times 2^{55555} - 5$.

Problem source : http://en.wikipedia.org/wiki/Prime_triplet

Solution

Red coloured numbers are divisible by 3. We will mark other composite numbers by blue.

Let's say p_n-6, p_n-2 and p_n form the last triplet prime set. We assume there is finite number of prime triplet set.

After p_n one of every triplet set should get divided by prime before it.

p_n-10 p_n-8 p_n-6
p_n-4 p_n-2 p_n
p_n+2 p_n+4 p_n+6
p_n+8 p_n+10 p_n+12
p_n+14 p_n+16 p_n+18
p_n+20 p_n+22 p_n+24
p_n+26 p_n+28 p_n+30
p_n+32 p_n+34 p_n+36
p_n+38 p_n+40 p_n+42
p_n+44 p_n+46 p_n+48
p_n+50 p_n+52 p_n+54
p_n+56 p_n+58 p_n+60
p_n+62 p_n+64 p_n+66
p_n+68 p_n+70 p_n+72
p_n+74 p_n+76 p_n+78
p_n+80 p_n+82 p_n+84
p_n+86 p_n+88 p_n+90
p_n+92 p_n+94 p_n+96
p_n+98 p_n+100 p_n+102
p_n+104 p_n+106 p_n+108

As we can see the primes occur in a regular manner if we mark the composite number as above. But primes don't have any regular pattern to generate.

Here is the contradiction. So this case cannot happen.

Again we form the table as below :

p_n-10 p_n-8 p_n-6
p_n-4 p_n-2 p_n
p_n+2 p_n+4 p_n+6
p_n+8 p_n+10 p_n+12
p_n+14 p_n+16 p_n+18
p_n+20 p_n+22 p_n+24
p_n+26 p_n+28 p_n+30
p_n+32 p_n+34 p_n+36
p_n+38 p_n+40 p_n+42
p_n+44 p_n+46 p_n+48
p_n+50 p_n+52 p_n+54
p_n+56 p_n+58 p_n+60
p_n+62 p_n+64 p_n+66
p_n+68 p_n+70 p_n+72
p_n+74 p_n+76 p_n+78
p_n+80 p_n+82 p_n+84
p_n+86 p_n+88 p_n+90
p_n+92 p_n+94 p_n+96
p_n+98 p_n+100 p_n+102
p_n+104 p_n+106 p_n+108

As we can see the primes occur in a regular manner if we mark the composite number as above. But primes don't have any regular pattern to generate.

Here is the contradiction. So this case cannot happen.

⇨ The composite number should occur in an awkward pattern

- ⇨ More primes are necessary to make them composite as for each prime occurrence of the multiples of the primes are regular but primes don't occur in regular pattern.

Again we form the table as below :

p_n-10 p_n-8 p_n-6
p_n-4 p_n-2 p_n
p_n+2 p_n+4 p_n+6
p_n+8 p_n+10 p_n+12
p_n+14 p_n+16 p_n+18
p_n+20 p_n+22 p_n+24
p_n+26 p_n+28 p_n+30
p_n+32 p_n+34 p_n+36
p_n+38 p_n+40 p_n+42
p_n+44 p_n+46 p_n+48
p_n+50 p_n+52 p_n+54
p_n+56 p_n+58 p_n+60
p_n+62 p_n+64 p_n+66
p_n+68 p_n+70 p_n+72
p_n+74 p_n+76 p_n+78
p_n+80 p_n+82 p_n+84
p_n+86 p_n+88 p_n+90
p_n+92 p_n+94 p_n+96
p_n+98 p_n+100 p_n+102
p_n+104 p_n+106 p_n+108

Not to form any prime triplet after p_n => there must be a composite number in each row.
Now, before p_n the number of primes is less than p_n.
Now, after p_n next multiple of p_n will occur in the same column is $6p_n$.
Between p_n and $6p_n$ there are p_n number of rows.
But before p_n there are less than p_n number of primes which can make composite one number of each row.

- ⇨ There will be primes left in a row.
- ⇨ Triplet prime set is there after p_n.
- ⇨ Triplet prime set is infinite.

Proved.
Corollary : *Twin primes are infinite as in the case of twin prime also each row should have at least one composite number.*

V. Problem 5 :

In **mathematics**, a superperfect number is a positive integer n that satisfies

$$\sigma^2(n) = \sigma(\sigma(n)) = 2n\,,$$

where σ is the divisor function. Superperfect numbers are a generalization of perfect numbers. The term was coined by Suryanarayana (1969).[1]

The first few superperfect numbers are

2, 4, 16, 64, 4096, 65536, 262144 (sequence A019279 in OEIS).

If n is an *even* superperfect number then n must be a power of 2, 2^k, such that $2^{k+1}-1$ is a Mersenne prime.

It is not known whether there are any *odd* superperfect numbers. An odd superperfect number n would have to be a square number such that either n or $\sigma(n)$ is divisible by at least three distinct primes. There are no odd superperfect numbers below $7\text{x}10^{24}$.

Problem source : http://en.wikipedia.org/wiki/Superperfect_number

Solution 5 :

Let N = (a^n)(b^m)...(k^y)}
Now the equations to satisfy the condition of existing a Superperfect numbers are :
$(1+a+a^2+....a\text{^}n)(1+b+b^2+...+b\text{^}m).....(1+k+k^2+...+k\text{^}y) = (p_1\text{^}a_1)(p_2\text{^}a_2)....(pj\text{^}aj)$ where j is suffix. (A)
and
$(1+p_1+p_1^2+...+p_1\text{^}a_1)(1+p_2+p_2^2+...+p_2\text{^}a_2)......(1+pj+pj^2+...+pj\text{^}aj) = 2\{(a\text{^}n)(b\text{^}m)...(k\text{^}y)\}$ (B)
Now if we divide equation (B) by 4 RHS ≡ 2 (mod 4)

- ⇨ LHS have only one odd power and rest are even otherwise LHS will be divided by 4.

Let's say a_1 is odd and rest powers are even.
Now dividing both sides of equation (A) RHS ≡ p_1 (mod 4)

- ⇨ RHS ≡ ± 1 (mod 4)

Now LHS of equation (A) should not be even.

- ⇨ All terms of LHS are odd.
- ⇨ n,m,...,k are even
- ⇨ N is a square number.

Let $N = \{(a\text{^}n)(b\text{^}m)...(k\text{^}y)\}^2 = (a\text{^}2n)(b\text{^}2m)....(k\text{^}2y)$
Now the equations to satisfy the condition of existing a Superperfect numbers are :
$(1+a+a^2+....a\text{^}2n)(1+b+b^2+...+b\text{^}2m).....(1+k+k^2+...+k\text{^}2y) = (p_1\ \text{^}a_1)(p_2\ \text{^}a_2)....(pj\text{^}aj)$ where j is suffix. (1)
and
$(1+p_1+p_1{}^2+...+p_1\ \text{^}a_1)(1+p_2+p_2{}^2+...+p_2\ \text{^}a_2)......(1+pj+pj^2+...+pj\text{^}aj) = 2\{(a\text{^}n)(b\text{^}m)...(k\text{^}y)\}^2$ (2)
Now, RHS of equation (2) ≡ 2 (mod 4) (as any odd square number is ≡ 1 (mod 4))

- ⇨ LHS should have only one odd power and rest are even power otherwise LHS will be divisible by 4.

Let's say a_1 is odd and rest are even.
Now RHS of equation (2) ≡ $\{(1+p_1)(a_1+1)/2\}\{(1+p_2)*(a_2/2)+1\}......\{(1+pj)*(aj/2)+1\}$

- ⇨ $1+p_1 = 2i$ (where i is odd).
- ⇨ $p_1 = 2(2i+1) - 1 = 4i+1$ (putting 2i+1 in place of i).
- ⇨ $p_1 \equiv 1 \pmod 4$

Now, let's say n, m, ..., k are odd and (1+a), (1+b), ... (1+k) are divisible by 2 and not 4.
if we take them even then the problem will go on increasing giving similar results. So generalization is not violated by taking the above assumption.
Now, 1+a+...+a^2n ≡ n(1+a) + 1 (mod 4)

- ⇨ 1+a+....+a^2n ≡ 3 (mod 4)
- ⇨ 1+b+....+b^2m ≡ 3 (mod 4) .... 1+k+....+k^2y ≡ 3 (mod 4)
- ⇨ LHS of equation (1) ≡ 3*3*...*3 ≡ 1 or 3 (mod 4)

RHS of equation (1) ≡ p_1 (mod 4) (as others are odd number square because a_1 other are even)

- ⇨ RHS of equation (1) ≡ 1 (mod 4) (as p_1 ≡ 1 (mod 4) shown above)
- ⇨ LHS has even number of terms otherwise LHS would be ≡ 3 (mod 4)

Now, n(1+a) +1 ≡ 3 (mod 4)

- ⇨ n(1+a) = 4w+2 form.

Here we will take w as odd. (if we take w as even then the problem will go on increasing giving the same solution. So generalization is not violated by taking w as odd)

- ⇨ n(1+a) +1 ≡ 7 (mod 8)
- ⇨ m(1+b) +1 ≡ 7 (mod 8) ...... y(1+k) ≡ 7 (mod 8)
- ⇨ LHS of equation (1) ≡ 7*7*....even terms ≡ 1 (mod 8)

But RHS of equation (1) ≡ p_1 (mod 8) (as other terms are odd square number ≡ 1 (mod 8))

- ⇨ RHS of equation (1) ≡ 5 (mod 8) as p_1 = 4i+1 and i is odd.

Here is the contradiction.

- ⇨ Odd superperfect number doesn't exist.

VI. PROBLEM 6 :

POLIGNAC'S CONJECTURE

In number theory, **Polignac's conjecture** was made by Alphonse de Polignac in 1849 and states:

> For any positive even number *n*, there are infinitely many prime gaps of size *n*. In other words: There are infinitely many cases of two consecutive prime numbers with difference *n*.[1]

The conjecture has not yet been proven or disproven for a given value of *n*. In 2013 an important breakthrough was made by Zhang Yitang who proved that there are infinitely many prime gaps of size *n* for some value of *n* < 70,000,000.[2]

For *n* = 2, it is the twin prime conjecture. For *n* = 4, it says there are infinitely many cousin primes (*p*, *p* + 4). For *n* = 6, it says there are infinitely many primes (*p*, *p* + 6) with no prime between *p* and *p* + 6.

Problem source : http://en.wikipedia.org/wiki/Polignac%27s_conjecture

Solution

For twin prime see the corollary of **Triplet Prime** problem. (End of Solution 4)

Now, we will prove there are infinitely many cousin primes.
Let, the series of cousin primes is finite.
Let, p_n – 6 and p_n -2 form the last cousin prime pair.
Red coloured numbers are divisible by 3. We will mark other composite numbers by blue.
Now we form the table as below :

p_n -4 p_n -2 p_n

p_n+2	p_n+4	p_n+6
p_n+8	p_n+10	p_n+12
p_n+14	p_n+16	p_n+18
p_n+20	p_n+22	p_n+24
p_n+26	p_n+28	p_n+30
p_n+32	p_n+34	p_n+36
p_n+38	p_n+40	p_n+42
p_n+44	p_n+46	p_n+48
p_n+50	p_n+52	p_n+54
p_n+56	p_n+58	p_n+60
p_n+62	p_n+64	p_n+66
p_n+68	p_n+70	p_n+72
p_n+74	p_n+76	p_n+78
p_n+80	p_n+82	p_n+84
p_n+86	p_n+88	p_n+90
p_n+92	p_n+94	p_n+96
p_n+98	p_n+100	p_n+102
p_n+104	p_n+106	p_n+108

The condition to satisfy that there is no cousin prime pair after p_n -2 is one of the two columns should be composite (except the column divisible by 3).

- ⇨ The primes occur in a regular fashion.
- ⇨ Here is the contradiction.

Now, to deny the case there needs to be composite number in the middle column in an irregular fashion.
Now, next p_n -2 occurs in the middle column after p_n -2 rows.

- ⇨ There are more than p_n -2 composite numbers before next p_n – 2 occurs in the table.
- ⇨ But before p_n -2 there are less than p_n -2 number of primes.
- ⇨ This case cannot happen.
- ⇨ There are infinite number of Cousin primes.

Proved.
The conclusion can be done in other way also :
All the number of third column is composite. But there is at least difference of 2 between any consecutive prime.

- ⇨ *There will be numbers left between the series of multiples of prime before p_n -2.*
- ⇨ *There will be Cousin prime generated after p_n -2.*

Here is the contradiction.

- ⇨ *There are infinite number of Cousin primes.*

Proved.
Now we will prove there are infinite number of sexy primes.
Let, the series of sexy primes is finite.
Let, p_n – 8 and p_n -2 form the last sexy prime pair.
Red coloured numbers are divisible by 3. We will mark other composite numbers by blue.
Now we form the table as below :

p_n -4 p_n -2 p_n

p_n+2	p_n+4	p_n+6
p_n+8	p_n+10	p_n+12
p_n+14	p_n+16	p_n+18
p_n+20	p_n+22	p_n+24
p_n+26	p_n+28	p_n+30
p_n+32	p_n+34	p_n+36
p_n+38	p_n+40	p_n+42
p_n+44	p_n+46	p_n+48
p_n+50	p_n+52	p_n+54
p_n+56	p_n+58	p_n+60
p_n+62	p_n+64	p_n+66
p_n+68	p_n+70	p_n+72
p_n+74	p_n+76	p_n+78

p_n+80 p_n+82 p_n+84
p_n+86 p_n+88 p_n+90
p_n+92 p_n+94 p_n+96
p_n+98 p_n+100 p_n+102
p_n+104 p_n+106 p_n+108

The condition to satisfy that there is no sexy prime pair after p_n -2 is alternate number in each column should be composite (except the column divisible by 3).

- ⇨ The primes occur in a regular fashion.
- ⇨ Here is the contradiction.

Now, to deny the case there needs to be more composite numbers.
Now, next p_n-2 occurs in the middle column after p_n-2 rows.

- ⇨ There are more than p_n-2 composite numbers before next $p_n - 2$ occurs in the table.
- ⇨ But before p_n -2 there are less than p_n -2 number of primes.
- ⇨ This case cannot happen.
- ⇨ There are infinite number of Sexy primes.

Proved.

VII. Problem 7

Are there infinitely many primes p such that $p - 1$ is a perfect square? In other words: Are there infinitely many primes of the form $n^2 + 1$? Ii is known as one of the **Landau's problems.**
Problem source : http://en.wikipedia.org/wiki/Landau%27s_problems

Solution 7 :

Now, n must be even. Because if n is odd then n^2+1 = even = composite.
Let the is the last prime of this form is $4k^2+1$.
So, we replace n by 2n.
The number is $4n^2 + 1$ where n is odd.
The equation is $4n^2 + 1 = (p_1\text{^}a_1)(p_2\text{^}a_2)...(pj\text{^}aj)$ where j is suffix.
Now, $4n^2+1 \equiv 1 \pmod 4$
Also, $4n^2+1 \equiv 5 \pmod 8$

- ⇨ RHS is of the form 4m+1 where m is odd.

Now, $4n^2 + 1 \equiv 5 \pmod{16}$
$4m + 1 \equiv \pm 4 \pmod{16}$

- ⇨ There will be cases when $4m+1\equiv -4 \pmod{16}$
- ⇨ A contradiction occurs.

Now, to avoid this contradiction m must be 4m+1 form.
Putting 4m+1 in place of m we get, 16m+5. Where m is odd.

- ⇨ On dividing 16m+5 by 32 gives 21 as remainder.

Now, $4n^2 + 1 \equiv 5 \pmod{32}$ [as $n^2 \equiv 1, -7, 9, -15 \pmod{32}$]
Here is the contradiction.

- ⇨ All $4n^2+1$ cannot be composite.
- ⇨ There are infinite number of primes which are of the form $n^2 +1$.

Proved.

VIII. Problem 8 :

Suppose there are k runners, all lined up at the start of a circular running track of length 1. They all start running at constant, but different, speeds.

The Lonely Runner conjecture states that for each runner, there will come a time when he or she will be a distance of at least 1/k along the track from every other runner.

The conjecture has been proved for small values of k (<=7).

The problem is to prove or disprove the conjecture for the general case, or for cases where k > 7.

Solution

Let's say, there are i number of participant in the race.
There velocities are V_1 , V_2, ... Vi where $V_2 > V_1 > .... >$ Vi
The j-th person completes the cycle in tj time (where j is suffix)
Let's suppose the velocities are in Arithmetic progression with common difference –d.
Now, let's take any participant j whose velocity is Vj (j is suffix)
Now Vj's previous and next participants are V(j-1) and V(j+1) in the first cycle.
Let's say there cannot be a lonely runner.

- ⇨ {V(j+1) – V(j-1)}*t_1 < 2/i (t_1 is the time when first person completes the cycle)

Let's say after time tk (k is suffix) first person (with velocity V_1) catches person j (with velocity Vj)
Now, (V_1 - Vj)tk = n where n is number of cycle after which first person catches j-th person. (k is suffix)
Upto this point the j-th person will be between (j+1)-th person and (j-1)-th person.
This time the distance between them becomes {V(j+1) – V(j-1)}*tk (where j+1, j-1, k are suffix)
As we have assumed no lonely runner this must be less than 2/i

- ⇨ {V(j+1) – V(j-1)}*tk < 2/i

Putting value of tk from above we get, [{V(j+1) – V(j-1)}*n]/(V_1 - Vj) < 2/i
Now, {V(j+1) – V(j-1)} = 2d and (V_1 - Vj) = (j-1)*d
Putting these values in above equation, we get,
2d*n/(j-1)*d < 2/i

- ⇨ n/(j-1) < 1/i
- ⇨ ni < (j-1)

Which is impossible.
Here is the contradiction.
Let's say the first participant meet again the j-th participant after 2tk time.
Now, the participant before and after the j-th participant be m-th and (m+1)-th.
{V(m+1) – Vm}2tk < 2/i (m+1 and m are suffix).

Putting value of tk we get, {V(m+1) – Vm}*n/{V_1 - Vj} < 1/i

- ⇨ d*n/(j-1)*d < 1/i
- ⇨ n*i < (j-1)

Which is impossible.
Here is the contradiction.

- ⇨ Every person will become lonely at some point of time.
- ⇨ Lonely Runner Conjecture is true when velocities are in AP.

Proved.

IX. Problem 9 :

The Riemann zeta-function ζ(*s*) is a function of a complex variable *s* defined by :

$$\zeta(s) = 1 + 1/2^s + 1/3^s + 1/4^s + \ldots$$

using analytical continuation for all complex $s \neq 1$.

The problem is to find the value of imaginary part of s for which *ζ(s) = 0.*

Solution

Te equation is : 1 + 1/2^s + 1/3^s + 1/4^s + ... = 0

- ⇨ (1 + 1/3^s + 1/5^s +......) + (1/2^s + 1/4^s + 1/6^s +......) = 0
- ⇨ (1 + 1/3^s + 1/5^s +......) + (1/2^s)(1 + 1/2^s + 1/3^s + 1/4^s +......) = 0
- ⇨ 1 + 1/3^s + 1/5^s + ... = 0 (as 1 + 1/2^s + 1/3^s + ... = 0)
- ⇨ 1 + e^ln(1/3)^s + e^ln(1/5)^s + ... = 0 (as a = e^lna where ln is natural logarithm to the base e)
- ⇨ 1 + e^{s*ln(1/3)} + e^{s*ln(1/5) + ... = 0
- ⇨ 1 + e^[(p+iq){ln(1/3)}] + e^[(p+iq){ln(1/5)}] + ... = 0 (s = p+iq where p, q real and i = √(-1))
- ⇨ 1 + [e^{p*ln(1/3)}]*[e^{iq*ln(1/3)}] + [e^{p*ln(1/5)}]*[e^{iq*ln(1/5)}] + ... = 0
- ⇨ 1 + [e^{ln(1/3)^p}]*[Cos{q*ln(1/3)} + iSin(q*ln(1/3)}] + ... = 0
- ⇨ 1 + {(1/3)^p}[Cos{q*ln(3)} - iSin(q*ln(3)] + {(1/5)^p}[Cos{q*ln(5)} - iSin(q*ln(5)] + ... = 0
- ⇨ [1 + {(1/3)^p}Cos{q*ln(3)} + {(1/5)^p}Cos{q*ln(5)} + ...] – i[{(1/3)^p}Sin{q*ln(3)} + {(1/5)^p}Sin{q*ln(5)} + ...] = 0
- ⇨ 1 + {(1/3)^p}Cos{q*ln(3)} + {(1/5)^p}Cos{q*ln(5)} + ... = 0 ....(1) and {(1/3)^p}Sin{q*ln(3)} + {(1/5)^p}Sin{q*ln(5)} + ... = 0 (Equating real and imaginary part from both sides) ...... (2)

[Another method of obtaining equation (1) and (2)
Let (1/m)^i = r*e^(iθ)
Taking natural logarithm to the base e on both sides we get,
i*ln(1/m) = ln(r) + iθ
Equating real and imaginary part from both sides we get,
ln(1/m) = θ & ln(r) = 0

- ⇨ 1/m = e^θ & r = 1
- ⇨ (1/m)^(p+iq) = e^(pθ+iqθ)
- ⇨ (1/m)^(p+iq) = {e^(pθ)}{e^(iqθ)}
- ⇨ (1/m)^(p+iq) = (1/m^p)[Cos{q*ln(m)} + iSin{q*ln(m)} (putting value of θ and e^(iqθ) = Cosθ + iSinθ)
- ⇨ (1/m)^(p+iq) = (1/m^p)Cos{q*ln(m)} + i(1/m^p)Sin{q*ln(m)}

Putting m = 3, 5, 7, ... and summing over all values, then equating the real and imaginary part from both sides, we get equation (1) and (2)]
Now, multiplying both sides of equation (1) by Sin{q*ln(1/3)} we get,
Sin{q*ln(1/3)} + {(1/3)^p}Sin{q*ln(1/3)}Cos{q*ln(3)} + {(1/5)^p}Sin{q*ln(1/3)}Cos{q*ln(5)} + ... = 0 .... (3)
Now multiplying both sides of equation (2) by Cos{q*ln(1/3) we get,
{(1/3)^p}Cos{q*ln(1/3)}Sin{q*ln(3)} + {(1/5)^p}Cos{q*ln(1/3)}Sin{q*ln(5)} + ... = 0 .... (4)
Now adding equation (3) and (4) we get,
Sin{q*ln(1/3)} + {(1/3)^p}[Sin{q*ln(1/3)}Cos{q*ln(3)} + Cos{q*ln(1/3)}Sin{q*ln(3)}] + ... = 0
Sin{q*ln(1/3)} + {(1/3)^p}Sin[q*{ln(1/3) + ln(3)}] + {(1/5)^p}Sin[q*{ln(1/3) + ln(5)}] + ... = 0
Sin{q*ln(1/3)} + 0 + {(1/5)^p}Sin{q*ln(5/3)} + {(1/7)^p}Sin{q*ln(7/3)} + ... = 0 .... (5)
Now adding equation (3) and (5) we get,
Sin{q*ln(1/3)} + {(1/3)^p}Sin{q*ln(3)} + {(1/5)^p}[Sin{q*ln(5) + Sin{q*ln(5/3)] + ... = 0

- ⇨ 2Sin{(q/2)*ln(1/3)Cos{(q/2)ln(1/3) + 2Sin{(q/2)ln(3)}Cos{(q/2)ln(3) + {(1/5)^p}*2Sin{(q/2)ln(25/3)}Cos{(q/2)ln(3) + ... = 0 (as SinA = 2Sin(A/2)Cos(A/2) and SinC + SinD = 2Sin{(C+D)/2}Cos{(C-D)/2})
- ⇨ 2Cos{(q/2)*ln(3)}*[Sin{(q/2)ln(1/3) + Sin{(q/2)ln(3) + Sin{(q/2)ln(25/3)} + Sin{(q/2)ln(49/3)} + ... = 0 (as Cos{(q/2)ln(1/3) = Cos{(q/2)ln(3) because Cos(-A) = CosA)
- ⇨ 2Cos{(q/2)*ln(3) = 0
- ⇨ Cos{(q/2)*ln(3) = 0
- ⇨ (q/2)ln(3) = nπ ± (π/2) (where n is integer)
- ⇨ q = (2nπ ± π)/ln(3) = (2n ± 1)π/ln(3)

Similarly by multiplying both sides of equation (1) by Sin{q*ln(1/5)} and equation (2) by Cos{q*ln(1/3)} then adding both the equation and following the steps as above we can prove,
Cos{(q/2)ln(5) = 0

- ⇨ q = (2n ± 1)π/ln(5)

Similarly it can be proved for any odd integer 2a+1, q = (2n±1)π/ln(2a+1) where a = 1, 2, 3, ...

Problem 10 :

In 1967, L. J. Lander, T. R. Parkin, and John Selfridge (LPS) conjectured that if

$$\sum_{i=1}^{n} a_i^k = \sum_{j=1}^{m} b_j^k$$

where $a_i \neq b_j$ are positive integers for all $1 \leq i \leq n$ and $1 \leq j \leq m$, then $m+n \geq k$

The problem is to prove the conjecture or find a counter-example

Problem source : http://en.wikipedia.org/wiki/Lander,_Parkin,_and_Selfridge_conjecture

Solution

Let, $m+n < k$

We will prove this with assumption $m < n < 2m$ because similarly can be proved for $md < n < m(d+1)$ where $d = [n/m]$ (where $[x]$ = greatest integer contained in x)

Let the ordered set of a1, a2,....an (n is suffix) = p1, p2, ..., pn (n is suffix) where $p_1 < p_2 < .... < p_n$

Similarly, ordered set of b1, b2...,bm (m is suffix) = q1, q2, ...qn where $q_1 < q_2 < .... < q_m$

Now applying AM ≥ GM on $p_1^k, p_2^k, ..., p_n^k$ we get,

$\{(p_1^k) + (p_2^k) + ... + (p_n^k)\}/n \geq (p_1 * p_2 * .... * p_n)^{(k/n)}$

- ⇨ $\{(p_1^k) + (p_2^k) + ... + (p_n^k)\}^{(n/k)} \geq \{n^{(n/k)}\}(p_1 * p_2 * .... * p_n)$

Similarly, $\{(q_1^k) + (q_2^k) + ... + (q_m^k)\}^{(m/k)} \geq \{m^{(m/k)}\}(q_1 * q_2 * ... * q_m)$

Now multiplying both the equations, we get,

$\{(q_1^k) + (q_2^k) + ... + (q_m^k)\}^{\{(m+n)/k\}} \geq \{n^{(n/k)}\}\{m^{(m/k)}\}((p_1 * p_2 * .... * p_n))(q_1 * q_2 * ... * q_m)$ (putting $p_1^k + p_2^k + ... + p_n^k = q_1^k + q_2^k + ... + q_m^k$)

- ⇨ $\{n^{(n/k)}\}\{m^{(m/k)}\}((p_1 * p_2 * .... * p_n))(q_1 * q_2 * ... * q_m) \leq \{(q_1^k) + (q_2^k) + ... + (q_m^k)\}^{\{(m+n)/k\}} < \{(q_1^k) + (q_2^k) + ... + (q_m^k)\}$ (as $m+n < k$ assumed)
- ⇨ $\{n^{(n/k)}\}\{m^{(m/k)}(p_n * q_1/q_m)(p_{(n-1)} * q_2/q_m).....(p_{(n-(m-2))} * q_{(m-1)}/q_m)(p_{(n-(m-1))}))...(p_n) < (q_1/q_m)^k + (q_2/q_m)^k + ... + (q_{(m-1)}/q_m)^k + 1$ [Dividing both sides by q_m^k and grouping the terms in LHS] ....(A)

Now, $p_n * q_1/q_m > 1$, $p_{(n-1)} * q_2/q_m > 1$, ..., $p_{(n-(m-1))} * q_{(m-1)}/q_m > 1$ (this is trivial)

Now, $q_m > q_i$ (where i, m are suffix and i = 1, 2, ..., m-1)

- ⇨ $1 > q_i/q_m$
- ⇨ $1 > (q_i/q_m)^{(k-1)}$
- ⇨ $q_i/q_m > (q_i/q_m)^k$ [Multiplying both sides by q_i/q_m]
- ⇨ $(p_n * q_1)/q_m > (q_1/q_m)^k$, $(p_{(n-1)} * q_2)/q_m > (q_2/q_m)^k$, ..., $(p_{(n-(m-2))} * q_{(m-1)})/q_m > (q_{(m-1)}/q_m)^k$
- ⇨ Equation (A) cannot hold true.
- ⇨ Our assumption was worng.
- ⇨ $m+n \geq k$
- ⇨ The conjecture is true.

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