Magic Square of Squares

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Abstract— **A n×n array of integers is called a magic square when all the rows, all the columns, across diagonals numbers add up to same sum. Here we will consider a 3×3 array. Magic square of squares is a 3×3 array whose all rows, all columns, across diagonals numbers are square number and also add up to same sum.**

Index Terms— **n×n array, integers, magic square**

I. Introduction

A square is magic if each of the rows, columns, and diagonals add up to the same total. So, for example, the square

2521	49	1465
289	1345	2401
1225	2641	169

is magic, since every row, column, and diagonal adds up to 4035. Of the nine entries, five (49, 169, 289, 1225, and 2401) are perfect squares.

The problem is to find a 3 by 3 magic square all of whose entries are distinct perfect squares, or prove that such a square cannot exist.

We will define the square.

Let the square entry be :

$A11^2$ $A12^2$ $A13^2$

$A21^2$ $A22^2$ $A23^2$

$A31^2$ $A32^2$ $A33^2$

Equations :

1) $A11^2 + A12^2 = A23^2 + A33^2$
2) $A21^2 + A23^2 = A12^2 + A32^2$
3) $A11^2 + A33^2 = A13^2 + A31^2$
4) $A11^2 + A21^2 + A31^2 = A12^2 + A22^2 + A32^2 = A13^2 + A23^2 + A33^2$
5) $A11^2 + A12^2 + A13^2 = A21^2 + A22^2 + A23^2 = A31^2 + A32^2 + A33^2$

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Definition :

Here we will define the acronyms used through the entire paper.

Now any of Aij^2 where i = 1,2,3 and j = 1,2,3 can be either odd or even.

We will write "e" for even and "o" for odd.
If any Aij is even it will be replaced by "e" and if odd it will be replaced by "o"

Assumption will be taken on A11, A12, A13, A23 and A33. Rest are dependent according to the above equations.

We will put a black "e" or "o" if it is assumed and we will put a red "e" or "o" if it is derived from equations.

We will follow this convention throughout this paper.

II. SOLUTION

Case 1 :

All Aij cannot be even otherwise the whole family will be divided by 4 until we get some odd. So, there is no question of considering this case.

Case 2 : (ANY TWO OF FIRST ROW IS ODD AND ANOTHER EVEN)

Let's say A23 = e and A33 = o

Case 2a : A11 = e ; A12 = e and A13 = o

e	e	o
		e
e	e	o

Explanation of derived even or odds :

$Odd^2 \equiv 1 \pmod 4$ and $even^2 \equiv 0 \pmod 4$

According to equation 3, $A31^2$ = even

According to equation 4, $A32^2$ = even

Now, column 3 is ≡ 2 (mod 4)

Whereas, column 1 or 2 is ≡1 or 0

Contradiction. So, this arrangement is not possible.

Case 2b : A11 =e, A12 =o , A13 = o

e o o

e e

e o

Explanations of derived even or odds :

According to equation 3, $A31^2 = e$

According to equation 4, $A21^2 = e$

Now, first column is ≡ 0 (mod 4)

Third column ≡ 1 (mod 4)

Contradiction. This arrangement is not possible.

Case 2c : A11 = o, A12 = e, A13 = o

o e o
e

o

e

o

Explanations of derived even or odds :

According to Equation 3, $A31^2$ = odd

According to equation 5, $A32^2 = e$

Now, whatever be A22 second column is ≡ 2 or 3 but third column is ≡1

Contradiction. This arrangement is not possible. Case 2d : A11 =o, A12 = =o. A13 = o

o o o

e

o

Here it is clear that third column is ≡ 2 (mod 4) whereas first row is ≡ 3 (mod 4) Contradiction. This arrangement is not possible.

Case 3 :

Fix A23 = even and A33 = odd.

Case 3a : A11 = e ; A12 = e and A13 = o

e e o

e

o

Here it is clear that first row is ≡1 (mod 4) whereas third column is ≡ 2 (mod 4)

Contradiction. This arrangement is not possible. Case 3b : A11 = e; A12 = 0, A13 = o

e o o

e

e o o

Explanations of the derived even and odds :

From equation 3, A31 = e
From equation 5, A32 = o

Now, first row is ≡ 2 (mod 4) whereas third row is ≡ 3 (mod 4)

Contradiction. This arrangement is not possible.

Case 3c : A11=o, A12 =e. A13 = o

o e o

e

o e o
Now whatever be the value of A22 second column is ≡ 1 or 0 (mod 4) whereas third column is ≡ 2

(mod 4)

Contradiction. This arrangement is not possible. Case 3d : A11 = o ; A12 = o; A13 =o

o o o

e o

It is clear that first row is ≡3 (mod 4) whereas third column is ≡ 2 (mod 4)

Contradiction. This arrangement is not possible.

Case 4 : A13 = even

Case 4a : A11 = e, A12 =e, A23 = e, A33 =o

e e e

e o

It is clear that first row is ≡ 0 (mod 4) whereas third column is ≡ 1 (mod 4)

Contradiction. This arrangement is not possible. Case 4b : A11 = e; A12 =e. A23 =o, A33 = e

e e e
o e

Similar conclusion as Case 4a.

Case 4c : A11 = e, A12 =o, A23 =e, A33 =o

e o e

e

o o

Explanation of derived even and odds : From equation 3, A31

= o

Whatever be the value of A32 third row is ≡ 2 or 3 (mod 4) whereas first row is ≡ 1 (mod 4)

Contradiction. So, this arrangement is not possible. Case 4d :

A11= e ; A12 =o, A23 =o, A33 = e

e o e

o

e o e

Explanations of the derived even and odds :

From equation 3, A31 = e

From equation 5, A32 = o

Now, whatever be the value of A22 second column is ≡ 2 or 3 (mod 4) whereas third column is ≡ 1

(mod 4).
Contradiction. This arrangement is not possible. Case 4e : A11 = e, A12 =o, A23 =o, A33 =o

e o e

o

o
It is clear that first row is ≡1 (mod 4) whereas third column is ≡ 2 (mod 4)
Contradiction. This case is not possible.
Case 4f : A11 =e, A12 =o, A23 = e, A33 = e
e o e

e

e

It is clear that first row is ≡ 1 (mod 4) whereas third column is ≡ 0 (mod 4)

Contradiction. This arrangement is not possible. Case 4g :

A11 =o, A12 =e, A13 = o, A33 =o

O e e

O

O

It is clear that first row is ≡ 1 (mod 4) whereas third column is ≡ 2 (mod 4)

Contradiction. This arrangement is not possible. Case 4h :

A11 =o, A12 =e, A23 = e, A33 = e

o e e

e

e

It is clear that first row is ≡ 1 (mod 4) whereas third column is ≡ 0 (mod 4)

Contradiction. So, this arrangement is not possible.

From the above cases we conclude that for any combination of independent variables A11, A12, A23, A33 the variable A13 doesn't exist.

Now, we will consider the last case if all are odd.

Now, any odd square number when divided by 3 gives remainder either 0 or 1. We will write 0 and 1 in place of Aij where i =1,2,3 and j = 1,2,3.

We will use red ink to show derived 0 or 1. We will follow this convention here.

Case A : Set $A31^2 \equiv 0$ (mod 3)

Case A1 : Set $A11^2 \equiv 0$, $A12^2 \equiv 0$, $A23^2 \equiv 0$, $A33^2 \equiv 1 \pmod 3$

0 0 0

0

1

It is clear that first row is $\equiv 0 \pmod 3$ whereas third column is $\equiv 1 \pmod 3$

Contradiction. This arrangement is not possible.

Case A2 : $A11^2 \equiv 0$, $A12^2 \equiv 0$, $A23^2 \equiv 1$, $A33^2 \equiv 0 \pmod 3$

0 0 0

1

0

Similar argument as Case A1.

Case A3 : $A11^2 \equiv 1$, $A12^2 \equiv 0$, $A23^2 \equiv 0$, $A33^2 \equiv 1 \pmod 3$

1 0 0

0

1

Now $A31^2 \equiv 0$ or 1. Whatever be the case Equation 3 doesn't satisfy.

Contradiction. This arrangement is not possible.

Case A4 : $A11^2 \equiv 0$, $A12^2 \equiv 1$, $A23^2 \equiv 0$, $A33^2 \equiv 1 \pmod 3$

0 1 0

0

1 1

According to equation 3, $A31^2 \equiv 1 \pmod 3$

Now whatever be the value of $A32^2$ equation 5 doesn't satisfy.

Contradiction. This arrangement is not possible.

Case A5 : $A11^2 \equiv 1$, $A12^2 \equiv 0$, $A23^2 \equiv 1$, $A33^2 \equiv 0 \pmod 3$

1 0 0

1

1 0

From equation 3, $A31^2 \equiv 1 \pmod 3$

Now, whatever be the value of $A21^2$ equation 4 doesn't satisfy.

Case A6 : $A11^2 \equiv 1$, $A12^2 \equiv 1$, $A23^2 \equiv 1$, $A33^2 \equiv 0 \pmod 3$

1 1 0

1

0

It is clear that first row is $\equiv 2 \pmod 3$ whereas third column is $\equiv 1 \pmod 3$

Contradiction. This arrangement is not possible.

Case A7 : $A11^2 \equiv 1$, $A12^2 \equiv 1$, $A23^2 \equiv 0$, $A33^2 \equiv 0 \pmod 3$

1 1 0

0

0

It is clear that first row is $\equiv 2 \pmod 3$ whereas third column is $\equiv 0 \pmod 3$

Contradiction. This arrangement is not possible.

Case A8 : $A11^2 \equiv 0$, $A12^2 \equiv 0$, $A23^2 \equiv 1$, $A33^2 \equiv 1 \pmod 3$

0 0 0

1

1

It is clear that first row is $\equiv 0 \pmod 3$ whereas third column is $\equiv 2 \pmod 3$

Contradiction. This arrangement is not possible.

Case B : Fix $A31^2 \equiv 1$

Case B1 : $A11^2 \equiv 0$, $A12^2 \equiv 0$, $A23^2 \equiv 0$, $A33^2 \equiv 1 \pmod 3$

0 0 1

0

1

It is clear that first row is $\equiv 1 \pmod 3$ whereas third column is $\equiv 2 \pmod 3$

Contradiction. This arrangement is not possible.

Case B2 : $A11^2 \equiv 0$, $A12^2 \equiv 0$, $A23^2 \equiv 1$, $A33^2 \equiv 0 \pmod 3$

0 0 1

1

0

It is clear that first row is ≡ 1 (mod 3) whereas third column is ≡ 2 (mod 3)

Contradiction. This arrangement is not possible.

Case B3 : $A11^2 \equiv 1$, $A12^2 \equiv 0$, $A23^2 \equiv 0$, $A33^2 \equiv 1 \pmod 3$

1 0 1

0

1 0 1

According to equation 3, $A31^2 \equiv 1 \pmod 3$

According to equation 5, $A32^2 \equiv 0 \pmod 3$

Now, whatever be the value of $A22^2$, second column will be ≡ 0 or 1 (mod 3) whereas third column is ≡ 2 (mod 3)

Contradiction. This arrangement is not possible.

Case B4 : $A11^2 \equiv 0$, $A12^2 \equiv 1$, $A23^2 \equiv 0$, $A33^2 \equiv 1 \pmod 3$

0 1 1

0

0 1

According to equation 3, $A31^2 \equiv 0 \pmod 3$

Now whatever be the value of $A21^2$ First column will be ≡ 0 or 1 (mod 3) whereas first row is ≡ 2
(mod 3).

Contradiction. This arrangement is not possible.

Case B5 : $A11^2 \equiv 1$, $A12^2 \equiv 0$, $A23^2 \equiv 1$, $A33^2 \equiv 0 \pmod 3$

1 0 1

1

0 0

According to equation 3, $A31^2 \equiv 0 \pmod 3$

Now, whatever be the value of $A32^2$, third row will be ≡ 0 or 1 (mod 3) whereas first row is ≡ 2 (mod
3).

Contradiction. This arrangement is not possible.
Case B6 : $A11^2 \equiv 1$, $A12^2 \equiv 1$, $A23^2 \equiv 1$, $A33^2 \equiv 0 \pmod 3$

1 1 1

1

0

It is clear that first row is ≡ 0 (mod 3) whereas third column is ≡ 2 (mod 3).

Contradiction. This arrangement is not possible.

Case B7 : $A11^2 \equiv 1$, $A12^2 \equiv 1$, $A23^2 \equiv 0$, $A33^2 \equiv 0 \pmod 3$

1 1 1

0

0

It is clear that first row is ≡0 (mod 3) whereas third column is ≡ 1 (mod 3)

Contradiction. This arrangement is not possible.

Case B8 : $A11^2 \equiv 0$, $A12^2 \equiv 0$, $A23^2 \equiv 1$, $A33^2 \equiv 1 \pmod 3$

0 0 1

1

1

It is clear that first row is ≡ 1 (mod 3) whereas third column is ≡ 0 (mod 3)

Contradiction. This arrangement is not possible.

Now, comes the interesting story. What if all numbers are ≡ 1

(mod 3). Then all numbers are ≡ 1 (mod 4)

So, the numbers are of the form 12n+1.

All numbers must be ≡0 or ±1 (mod 5)

So, the last digit of the numbers can be 9, 1 or 5.

So, sum of any row or any column or across the diagonals should be divisible by 5.

Let's adorn the number in the 3×3 array with last digit.

Best combination can be 9 1 5

1 5 9

5 9 1

Now, $A13^2 + A31^2 \equiv 50 \pmod{100}$

So, $(A11^2 + A33^2)$ & $(A12^2 + A32^2)$ & $(A21^2 + A23^2)$ must be ≡ 50 (mod 100)

The tenth digit combination can be (9,5) ; (8,6) ; (7,7) ; (2,2) and (3,1)

From these only (8,6) and (2,2) is possible combination. Other cases are not applicable.

So, the last two digits of the number is of the form 89, 81, 69, 61, 29, 21.

The pairs can be (29, 21); (89,61); (81,69)

If we go on squaring the odd numbers from 9 to 100, we get as last two digits as following :

21 -> 11^2, 39^2, 61^2, 89^2

29 -> 23^2, 27^2, 73^2, 77^2

89 -> 17^2,33^2, 83^2, 133^2

61 -> 19^2, 31^2, 81^2, 131^2

81 -> 9^2, 41^2, 59^2, 91^2

69 -> 13^2, 37^2, 63^2, 87^2

We see a pattern that 21 and 29 comes with a difference of 12 whereas 89 and 61 comes as a difference of 2 and 81 and 61 comes as a difference of 4.

We also can note that 21 numbers occur in the difference of 28,22,28. 29 series occur on the difference of 4, 46,4. 89 series occur in the difference of 16, 50, 50. 61 series occur in the difference of 12, 50, 50. 81 series occur in the difference of 32,18,32. 69 series occur in the difference of 24,26,24.

So, we see that the series sum of every two consecutive difference is 50 except for 89 and 61.

Now, we see that 61 and 89 comes in a series of difference 50.

So, we can say any number of the series are $(31+50n)^2$ and $(33+50m)^2$

Now, if we take numbers from 21 and 29 series one after another then it will be, $(11+50p)^2$ and $(23+50q)^2$

Now, $(31+50n)^2 + (33+50m)^2 = (11+50p)^2 + (23+50q)^2$

$\Rightarrow 14 + (31n + 33m) – (11p + 23q) + 25 (m^2 + n^2 - p^2 - q^2) = 0$ ....equation (1)

If we take the other series of 21 and 29 we get,

$(31+50n)^2 + (33+50m)^2 = (39+50p)^2 + (27+50q)^2$

$\Rightarrow 2 + (39p + 27q) – (31m + 33n) + 25(p^2+q^2-m^2-n^2) = 0$ ....equation (2)

Now, equation (1) or (2) should have a solution to hold magic square of squares.

Similarly, we can form equation for 81 and 69.

III. Conclusion

We have found the last two digits of the square numbers of Magic square. But we can’t conclude that one doesn’t exist.

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